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AN INTRODUCTION TO VECTOR ANALYSIS

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for

PHYSICISTS AND ENGINEERS

by

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PREFACE TO THIRD EDITION

In preparing this third edition it has not been considered necessary to make any change in the lay-out of the text, the general form of which seems to have served the needs of the physicists and engineers for whom the book was written.

Opportunity has been taken, however, to add some further notes on vector notation (p. 117), on the more formal aspect of the del operator (p. 113) and on Maxwell's equations (p. 118). At the same time a number of minor alterations and corrections have been made. I am indebted to Professor S. Chapman, F.R.S., and to Professor G. Cook, F.R.S., for kindly criticisms and helpful suggestions which have enabled me to make the text clearer and more accurate. For certain historical facts I am indebted to the late Dr. John McWhan.

GLASGOW
Jan. 1945.

PREFACE

Vector Analysis is the natural means of expression for the three-dimensional problems of physics and engineering, because its conciseness and freedom from mathematical detail enable the relationships between the various physical quantities to be kept clearly in view. Since the pioneer work of Gibbs and of Heaviside an increasing number of text-books and scientific papers on physical and technical subjects have made use of vector methods, until it has now become almost essential for any advanced worker in these sciences to have some knowledge of vector analysis. Much good work can be done with the aid of a very few elementary principles. It is the object of this monograph to give an introduction to these principles and to explain them from a physical standpoint, so that they may be easily available to the busy physicist or engineer approaching the subject for the first time. Such workers are usually so much occupied by their major task as to lack the time necessary to enable them to seek out such principles as they need to use from the more comprehensive treatises which aim at mathematical completeness. The monograph is not, therefore, intended for the reader with purely mathematical interests, whose more rigorous and systematic requirements are fully satisfied elsewhere.

For these reasons the outlook adopted is almost entirely physical; geometrical matters and questions of an exclusively mathematical interest are limited to essentials. Formal proofs of invariance and conditions of continuity in vector processes are replaced by an appeal to physical intuition. Purely analytical topics of an advanced kind, such as Green's theorem, are justifiably omitted from such an introductory treatment as this monograph aims to present. The reader who later wishes to amend his knowledge in these and other respects will find ample material in the standard books listed in the Bibliography.

The reader's attention is drawn in particular to two features. First, the use of trimetric projection by Gough's method (*Engineering*, vol. 143, p. 458, 1937) for certain of the three-dimensional diagrams, a purpose for which this method is admirably suited. Second, the inclusion of a chapter giving a brief sketch of the elementary properties of tensors and dyadics in their relation to vectors. This is a subject which usually puzzles and often repels physics and engineering students because of the abstract mathematical way in which it is generally brought to their notice.

The monograph is based on a course of lectures given a few years ago to post-graduate electrical engineering students in the Polytechnic Institute of Brooklyn, New York. I am particularly grateful to my colleague Dr. A. J. Small for his valued assistance in reading the manuscript and proofs, and for numerous suggestions.

GLASGOW
June 1938

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AN INTRODUCTION TO VECTOR ANALYSIS

CHAPTER I

DEFINITIONS. ELEMENTS OF VECTOR ALGEBRA

1. Scalar and Vector Quantities. Physical quantities are divided into two main classes, each with characteristic properties and an appropriate algebra.

Scalar quantities have magnitude only and do not involve direction. Typical scalar quantities are mass, volume, density, temperature, electric potential, charge, &c. The complete specification of a scalar quantity requires (i) a unit of the same kind and (ii) a number stating how many times the unit is contained in the quantity. For example, to express the mass of a given body we require to know whether the unit is the pound, the ton, the gramme, &c., and also how many of the chosen units represent the given mass. Scalar quantities are manipulated by applying the rules of ordinary algebra to their numerical magnitudes; for this reason the algebra of ordinary positive and negative numbers is often called *scalar algebra* and the numbers themselves *scalars*.

Vector quantities have magnitude and direction. Familiar examples are displacement, force, velocity, acceleration, stress, electric force, magnetic induction, &c. A vector quantity requires for its specification (i) a unit of the same kind, disregarding direction, (ii) a number giving the magnitude of the quantity in terms of this unit and (iii) a statement of direction. For example, the velocity of a moving body is stated by saying (i) that the unit is miles per hour, kilometres per second, &c.; (ii) how many of the chosen units express the magnitude of the velocity, and (iii) the sense in which the velocity is directed, e.g. due north. The combination of conditions (ii) and (iii) constitutes the geometric conception of a directed magnitude or *vector*, quite independent of the kind of vector quantity specified by the unit. The directional element will prevent the manipulation of vectors by the simple numerical algebra applying to scalars; it is essential, therefore, to devise a *vector algebra* by means of which vectors may be handled in a way consistent with the physical problems in which vector quantities occur. The laws of vector algebra differ in several important respects from those of scalar algebra, as will be seen later.

Vector quantities, though the simplest, are not the only non-scalar quantities found in physics and engineering. In the theory of inhomogeneous strain, in the transformations of space involved in the theory of relativity and in the general theory of electromagnetic machinery, to mention only a few examples, more complex non-scalars appear. These are the linear vector function, the dyadic and the tensor. Each has its own algebra, differing radically from the scalar laws and more general in form than vector algebra; indeed, the particular quantity is characterized by its algebra. To manipulate these quantities with conciseness and ease, higher algebraical methods, multiple algebras and the theory of matrices are widely used. In this elementary monograph it is not possible to do more than sketch the simplest properties of these higher non-scalars and their relation to ordinary vectors; the reader will find further developments in the special treatises referred to in the Bibliography.

Scalars are manipulated by six elementary operations, namely, addition and subtraction, multiplication and division, involution and evolution. In each pair the second operation is the inverse of the first. Also, multiplication is extended addition, and involution (i.e. raising a number to a power) is extended multiplication; likewise, subtraction, division and evolution are successively extended inverse operations. All these operations are performed by means of certain laws, which will now be stated for addition and multiplication; they are readily extended to the other operations and the reader will find formal proofs in any good text-book of algebra:

(i) *The law of association* can be written as

$$a + (b + c) = (a + b) + c,$$

and

$$a \times (b \times c) = (a \times b) \times c;$$

i.e. the result of adding a to the sum of b and c is the same as adding the sum of a and b to c; with a similar statement for products. In other words, brackets are unnecessary in continuous sums and products.

(ii) *The law of commutation* which states that additions and multiplications can be made in any order; i.e.

$$a + b = b + a,$$

and

$$a \times b = b \times a.$$

(iii) *The law of distribution*, stating that in compound expressions involving multiplication and addition the result is equivalent to the sum of individual products taken term by term, as in the rules

$$a \times (b + c) = a \times b + a \times c,$$

and

$$(a + b) \times c = a \times c + b \times c.$$

The three laws are extended to the inverse operations by the rule of signs, and additional rules for the manipulation of indices in involution are also provided. Finally, a fourth law, fundamental in the logical discussion of inverse operations, is now usually stated, namely, if $a \times b = 0$, then either $a = 0$ or $b = 0$.

While these scalar operations and their laws are familiar to all as the working processes of arithmetic and algebra, so familiar indeed as to be regarded as self-evident, it is quite essential to state them in this formal way since they characterize scalar algebra. The essential feature of the algebras of vectors, tensors, matrices, &c., is that they all violate in some way the commutative law for multiplication and do not necessarily satisfy the condition that either factor in a product is zero if the result is zero. The physical meaning of these differences between scalar and vector algebra will be explained later.

2. Graphical Representation of Vectors. Since a vector is the result of abstracting magnitude and direction from a vector quantity, independently of the nature of the quantity concerned, we may represent the vector graphically by a line *OA* pointing in the direction from *O* to *A*, as in Fig. 1. The magnitude of the vector is given to a

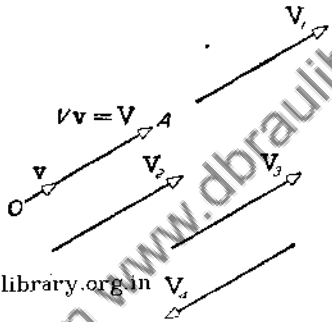


FIG. 1.—Definition of Equal Vectors

convenient scale by the length of the line; the direction in space is indicated by an arrow-head marked on the line. In the algebra of such vectors it is the custom to distinguish them by the use of a distinctive symbol. Thus, the vector *OA* is denoted by the Clarendon letter **V**; its scalar magnitude is stated by the corresponding italic letter *V*. A vector is unchanged by pure displacement; it follows, therefore, that all parallel vectors of the same length and direction are equal; a vector is not localized. Thus in Fig. 1 the vectors $\mathbf{V}$, $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ are all equal, a fact stated by the relation

$$\mathbf{V} = \mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}_3,$$

in which the meaning of the sign = is extended to include equality of size and similarity of direction. Reversing the arrow-head changes the sign of a vector, such as $\mathbf{V}_4$ in Fig. 1; then

$$\mathbf{V}_4 = -\mathbf{V}$$

indicates equality of size and opposition of direction.

This definition restricts the meaning of a vector to the representation of size and direction only. Consequently, a vector cannot represent *completely* any vector quantity the effect of which is changed by simple displacement; many quantities are of this kind, most notably force. It is clearly insufficient to specify a force by its magnitude and direction alone, i.e. by its vector. If the force acts on a rigid body the *line* of action must also be stated, since a change in the line of action alters the torque acting on the body. In the case of a deformable body the *point* of action of a force must be specified. These examples are given as a warning to the reader not to be tempted to attribute to a vector any properties, other than those of size and direction, which properly belong to the quantity that the vector is representing to scale merely in these particulars. In other words, vector analysis must always be used with an eye upon the physical conditions of the problem to which it is applied.

It is geometrically obvious that multiplication of a vector by a scalar factor S results in a vector S times the size of the first and in the same direction. In particular, if $\mathbf{v}$ be a vector of unit size in the direction of $\mathbf{V}$,

$$\mathbf{V} = V\mathbf{v}. \qquad (1.1)$$

Distinction is often made, particularly in more advanced work, between two essentially different classes of quantities represented by vectors. The first class includes quantities such as force, displacement, velocity, &c., in which the vector is drawn in the direction of the quantity concerned; mere linear action in a particular direction is involved and the vector is called a *polar vector*. The second class includes quantities such as angular velocity, angular acceleration, &c., in which rotary action of some kind takes place about an axis. Here the vector is drawn parallel to the axis about which the quantity acts; the length of the vector gives the magnitude of the quantity. The direction of the vector is conventionally fixed by the rule of the right-handed screw, i.e. when sighting along the direction shown by the arrow-head, rotation is taken as clockwise. Such vectors are named *axial vectors*. Th distinction is a physical one, there being little diff ence in the mathematical treatment of the two classes quantities. Indeed, it is difficult in many cases, particu larly in dynamics and electromagnetic theory, to assig vector quantities with certainty to classes in this way.

3. Addition and Subtraction of Vectors. Conside two vectors **A** and **B**, shown in Fig. 2(*a*), representing fo

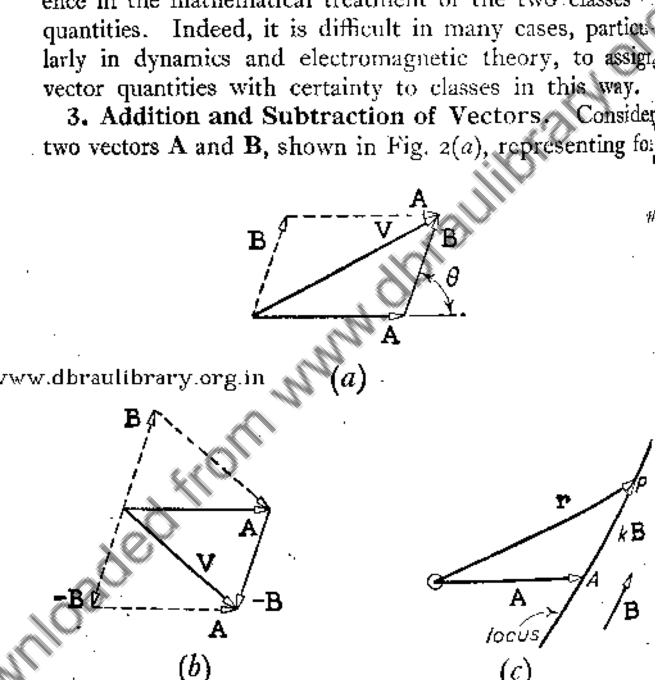


FIG. 2.—(*a*) Vector Addition; (*b*) Vector Subtraction; (*c*) Vector Equation of a Line

example two successive displacements of a point. Their joint effect, sum or resultant is obtained by setting off the vector **B** at the end of **A** and drawing the vector **V** joining the beginning of **A** to the end of **B**. The same result is obtained by starting with **B** and attaching **A** to it in a similar way, as shown by the dotted lines. Then

$$\mathbf{V} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \quad . \quad . \quad . \quad . \quad (1.2)$$

the + sign being understood to mean addition in this geometric sense. Thus the sum of two vectors is the diagonal of the parallelogram of which the vectors are the sides; such a sum is commutative, i.e. independent of which vector is taken first. Subtraction of one vector from another immediately follows; the vector **B** to be subtracted is reversed and the rule for addition applied, as illustrated by Fig. 2(*b*), in which $\mathbf{V} = \mathbf{A} + (-\mathbf{B}) = \mathbf{A} - \mathbf{B}$.

A useful geometrical application is shown in Fig. 2(*c*), where **A** and **B** are given vectors. If k is a variable parameter ranging between $+\infty$ and $-\infty$, the sum of **A** and $k\mathbf{B}$ is the radius vector from O, namely,

$$\mathbf{r} = \mathbf{A} + k\mathbf{B}.$$

As k varies, the locus of P will be a straight line through A parallel to **B**, and the expression is the vector form of the equation for a straight line.

The reader must not suppose that *all* quantities that may be represented by a directed line are *necessarily* vector quantities; the crucial test is whether they follow the parallelogram law of addition or not. For example, finite rotation of a rigid body about an axis can be represented by a line, of a length proportional to the angle of rotation, drawn in the direction of the axis. A second finite rotation about another axis inclined to the first may be similarly represented. But their resultant effect cannot be found by compounding the two finite rotations by the law for the vector sum, as the reader may easily verify in a simple case such as the motion of a ball on a plane. Finite rotations are not, therefore, capable of representation by vectors; they require the use of tensors and have much more complex laws of manipulation. Infinitesimal rotations, angular velocities and accelerations are, on the other hand, vector quantities (see p. 46).

With more than two vectors, such as **A**, **B**, **C** in Fig. 3, the sum is obtained by laying off the vectors successively end to end; then the sum is the closing side **V** of the polygon of which the other sides are **A**, **B** and **C**. But it is geometrically obvious that we could start by adding any

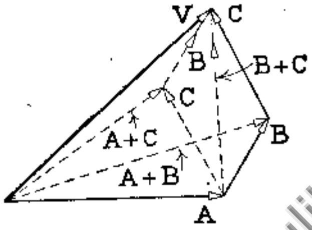


FIG. 3.—Law of Association for Vector Addition

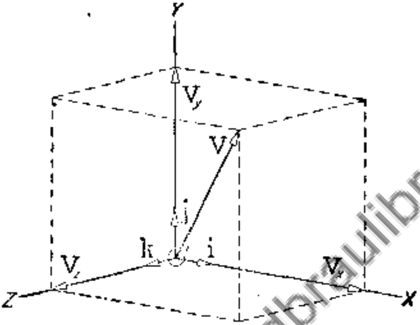
pair of vectors by the parallelogram rule, and to their resultant add the third vector. Hence

$$\mathbf{V}=\mathbf{A}+\mathbf{B}+\mathbf{C}=(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{C})+\mathbf{B}. \quad (1.3)$$

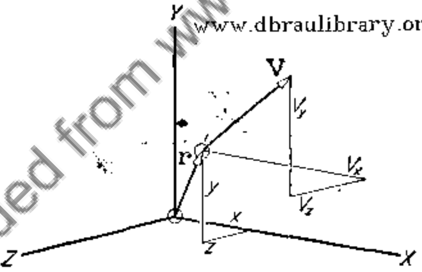
The sum of any number of vectors is, therefore, associative, i.e. the vectors may be added in any desired manner. The vectors are not necessarily all in the same plane; this is more clearly shown by Fig. 5.

4. Components of a Vector. By reversing the process of addition it is clear that any vector **V** can be decomposed into the sum of n vectors, of which $n-1$ are arbitrary and the last one closes the polygon. In general the vectors are not coplanar and the polygon is not a plane figure.

The most useful instance is the decomposition of a vector into component vectors along the three orthogonal axes of cartesian co-ordinates. Fig. 4(a) shows the usual right-handed system, in which the relative positive directions along the axes are chosen such that if $O'X$ is turned toward $O'Y$ about $O'Z$ through the smaller angle, a right-handed screw would advance along the positive direction of $O'Z$; similar relations hold for the axes $O'Y$ and $O'X$, always keeping X, Y and Z in cyclic order. Let the origin O' be one extremity of a vector **V**; draw a rectangular parallelepiped with the three edges which meet at O'



(a)



(b)

FIG. 4.—Cartesian Components of a Vector

lying along the axes and such that **V** is the diagonal from O' through the solid figure. If $\mathbf{V}_x$, $\mathbf{V}_y$, $\mathbf{V}_z$ are the vector intercepts along the axes of X, Y and Z respectively, then

$$\mathbf{V} = \mathbf{V}_x + \mathbf{V}_y + \mathbf{V}_z. \quad . \quad . \quad . \quad . \quad (1.4)$$

Now let **i**, **j**, **k** denote vectors of unit magnitude along the axes of X, Y and Z respectively; these are the *unit vectors* along the axes and are frequently used in later work. Then if V_x, V_y, V_z are the sizes of the vectors $\mathbf{V}_x$, $\mathbf{V}_y$, $\mathbf{V}_z$,

$$\mathbf{V}_x = V_x\mathbf{i},\ \mathbf{V}_y = V_y\mathbf{j} \text{ and } \mathbf{V}_z = V_z\mathbf{k},$$

from Equation (1.1); whence

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}. \quad . \quad . \quad . \quad . \quad (1.5)$$

In discussing the properties of a vector field we are concerned with the magnitude and direction of a vector quantity at every point in space. In rectangular co-ordinates with origin O let a point in space, such as O' in Fig. 4(*b*), be specified by the radius vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Since a vector is not affected by parallel displacement, Equation (1.5) is still the relation between the components of a vector and the vector itself. Note, however, that as we are dealing with a vector field, **V** is a function of position; that is, **V** and its components V_x, V_y, V_z are functions of the variables x, y, z specifying the position of O' relative to O.

Vector quantities and their vectors are physically independent of any system of co-ordinates by means of which they may be expressed; they are said to be *invariant*. Co-ordinate axes are unnecessary, therefore, in vector analysis. The resolution of a vector into components, given by Equation 1.5, is the connecting link between vector notation and that of ordinary co-ordinate geometry with which the reader is already familiar. Since a vector has three axial components, any vector equation relating vectors is the equivalent of three scalar cartesian equations relating their components. This conciseness is the great advantage of vector methods in three-dimensional problems. The fact that vector quantities are expressed directly by a notation natural to themselves, instead of by a long and artificial system of scalar relationships, enables their essential physical nature to be kept clearly in view. The resolution into cartesian components is often useful in the proof of theorems used in vector analysis; in the words of Heaviside, 'When in doubt and difficulty, fly to **i**, **j**, **k**', advice we shall frequently follow.

As a simple example find the sum of vectors **A**, **B**, **C**. In terms of their components,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},\ \mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k},$$
$$\mathbf{C} = C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k}.$$

The components along the X axis add directly; likewise those along the Y and Z axes. Then, as shown in Fig. 5,

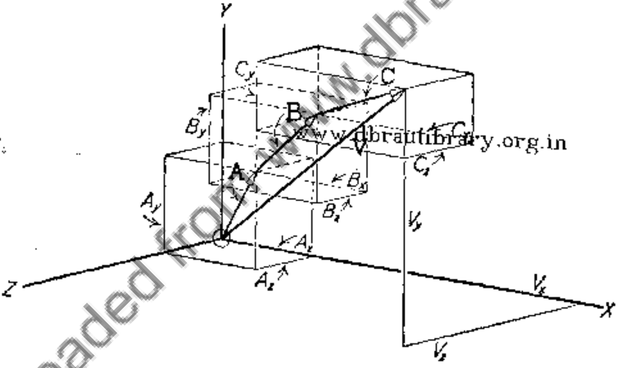


FIG. 5.—Cartesian Components of Vector Addition

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k} = \mathbf{A} + \mathbf{B} + \mathbf{C}$$
$$= (A_x + B_x + C_x)\mathbf{i} + (A_y + B_y + C_y)\mathbf{j}$$
$$+ (A_z + B_z + C_z)\mathbf{k} \quad . \quad . \quad (1.6)$$

Since the scalar magnitudes in the brackets obey the ordinary laws of algebra, it follows that vector sums (and differences) obey these laws also. The component of the resultant in any axial direction is the sum of the individual components in that direction; as is geometrically obvious from Fig. 5,

$$V_x = A_x + B_x + C_x,\ V_y = A_y + B_y + C_y \text{ and }$$
$$V_z = A_z + B_z + C_z.$$

5. Scalar and Vector Fields. A physical quantity can be expressed as a continuous function of the position of a point in a region of space; such a function is called a *point-function* and the region in which it specifies the physical quantity is known as a *field*. Fields are of two main kinds, scalar and vector, according to the nature of the quantity concerned.

A typical *scalar field*, such as the distribution of temperature, density, electric potential or of any other non-directed quantity, is represented by a continuous scalar function giving the value of the quantity at each point. Such a function does not undergo any abrupt change of magnitude in passing from any point to another close to it, a condition satisfied in all practical cases. The field can be mapped graphically by a series of surfaces—such as isothermal, equi-density or equipotential surfaces—upon each of which the scalar has a definite constant value. Such surfaces, called *equal or level surfaces*, are conveniently chosen so that in passing from one to the next a constant arbitrary difference is made between the scalars which characterize them. It is evident that the level surfaces must lie one within the other and cannot cut; for if two such surfaces could intersect, the scalar values corresponding to both must hold along their common line, which is contrary to our definition. Hence scalar point-functions are single valued or *uniform* at every point.

A typical *vector field*, such as the distribution of velocity in a fluid or of electric or magnetic field strength, is represented at every point by a continuous vector function. At any given point the function is specified by a vector of definite magnitude and direction, both of which change continuously from point to point throughout the field region. Starting at any arbitrary place, proceed an infinitesimal distance in the direction of the vector at that place, arriving at a closely-neighbouring point. Proceeding thence in a similar way, we shall trace out a curved line, the tangent to which at any point gives the direction of the vector thereat; such a curve is a *vector line, line of flow* or *flux line*. To represent the magnitude of the vector, at any point on a flux line draw a very small surface perpendicular thereto and choose a number of points per unit area upon this surface numerically equal to the magnitude of the vector. Through each of these points flux lines can be drawn. The field is then mapped out by flux lines. The direction of the lines is that of the vector function; their density, represented by the number of them crossing per unit area perpendicular to their direction, is a measure of the magnitude of the vector. It is clear that lines of flow cannot intersect, since this would involve indefinite direction of the vector at the point where they cut; vector point-functions must also be single valued at every point.

The physical properties of scalar and vector fields will be considered in greater detail in Chapters VII and VIII respectively.

CHAPTER II

THE PRODUCTS OF VECTORS

1. General. The ordinary idea of a product in scalar algebra, the mere multiplication of a scalar magnitude, cannot apply to vectors because of their directional properties; nor is it possible to decide by deductive reasoning what form the product of two vectors should take. Since vectors have their origin in physical problems, definitions for the products of vectors must be devised that will be consistent with the way in which such products occur in applications to physical science.

As a simple example consider vectors which represent a force **F** and a linear displacement **d**, their magnitudes being F and d and the angle between their directions θ. Products of these two quantities occur in two ways. First, the work done by the force is $Fd \cos \theta$, a scalar quantity, known as the *scalar product* of **F** and **d**. The couple exerted by the force has a magnitude $Fd \sin \theta$ and acts about an axis perpendicular to the plane containing **F** and **d**. It is shown by a vector, called the *vector product* of **F** and **d**, drawn along the axis of the couple in a conventional sense to be defined later. We are led, therefore, to define two sorts of products, namely, the scalar product and the vector product.

2. The Scalar Product. The scalar product of two vectors **A** and **B** is defined as the product of the magnitudes of the vectors and the cosine of the angle between their directions. As shown by Fig. 6, this is the same as the product of the size of one vector with the component of the other in the direction of the first, and this process is clearly commutative, since $\cos\theta = \cos(-\theta)$, i.e. independent of the order of the factors. The scalar product will

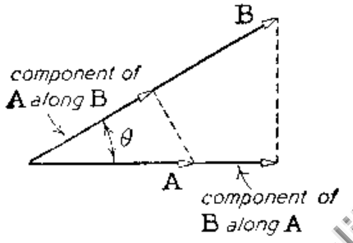


FIG. 6.—Scalar Product of Two Vectors

be denoted by interposing a dot between the vectors; then

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} = AB\cos\theta. \quad . \quad . \quad . \quad (2.1)$$

Thus, by definition, the scalar product is commutative. On p. 3 it has been pointed out that the product of two scalars vanishes when either is zero. The scalar product of vectors also satisfies this condition, but will vanish in addition when the vectors are at right angles. This is the first particular in which the laws of vector algebra differ from those of scalar algebra. When two vectors are perpendicular, therefore,

$$\mathbf{A}\cdot\mathbf{B} = 0, \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

and when they are parallel,

$$\mathbf{A}\cdot\mathbf{B} = AB. \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

If $\mathbf{B} = \mathbf{A}$, the scalar product of a vector with itself, called its *self-product*, is

$$\mathbf{A}\cdot\mathbf{A} = \mathbf{A}^2 = A^2; \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

hence to find the size of a vector calculate the square root of its self product.

Applying these properties to the unit vectors **i**, **j**, **k**, which are mutually perpendicular,

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0, \quad . \quad . \quad . \quad . \quad (2.5)$$

and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1. \quad . \quad . \quad . \quad . \quad (2.6)$$

Now examine the scalar product of a vector **A** with the sum of two others, **B** and **C**, illustrated in Fig. 7(*a*); the

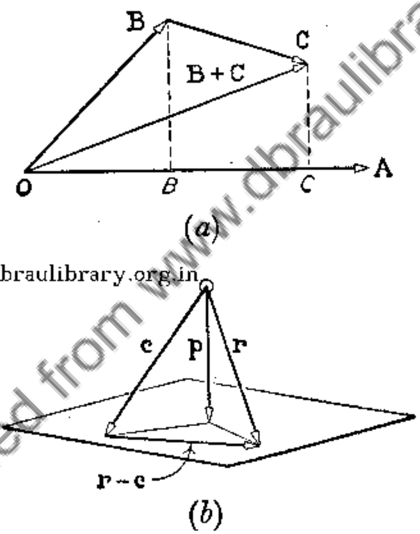


FIG. 7.—(*a*) Law of Distribution for Scalar Products; (*b*) Vector Equation of a Plane

three vectors are not necessarily in the same plane. Since the resolute of **B** + **C** on **A** is equal to the sum of the resolutes of **B** and **C**, i.e. $OC = OB + BC$,

$$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = A(OC) = A(OB + BC) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}. \quad (2.7)$$

Hence the scalar product is distributive for addition, as is geometrically obvious. By an extension of this process it is easy to show that

$$(\mathbf{A}+\mathbf{B}+\ldots)\cdot(\mathbf{N}+\mathbf{O}+\ldots) =$$
$$\mathbf{A}\cdot\mathbf{N}+\mathbf{A}\cdot\mathbf{O}+\ldots \quad \mathbf{B}\cdot\mathbf{N}+\mathbf{B}\cdot\mathbf{O}+\ldots \quad +\ldots \quad . \quad . \quad (2.8)$$

A particular case is of geometrical interest. In Fig. 7(*b*) let $\mathbf{p}$ be the perpendicular from O upon a plane which passes through the extremity of a given vector $\mathbf{c}$. If $\mathbf{r}$ is the radius vector to any other point in the plane, then it is obvious that the resolutes of $\mathbf{c}$ and $\mathbf{r}$ upon $\mathbf{p}$ must be equal, i.e.

$$\mathbf{p}\cdot\mathbf{r} = \mathbf{p}\cdot\mathbf{c} \text{ or } \mathbf{p}\cdot(\mathbf{r}-\mathbf{c}) = 0$$

is the equation of a plane perpendicular to $\mathbf{p}$ through the end of $\mathbf{c}$. From Equation 2.2, $(\mathbf{r}-\mathbf{c})$ must lie in the plane normally to $\mathbf{p}$, as is obvious.

Referring again to Fig. 2(*a*), the scalar product enables the usual formula for the resultant of two vectors to be readily found. Their sum is

$$\mathbf{V} = \mathbf{A} + \mathbf{B}.$$

Take the self product on both sides; then from Equations 2.4 and 2.8,

$$\mathbf{V}\cdot\mathbf{V} = \mathbf{V}^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B},$$

or

$$V^2 = A^2 + 2AB\cos\theta + B^2.$$

The rectangular components of a vector are found at once by taking its scalar product with the unit vectors, as is evident from Fig. 4; then

$$\mathbf{V}\cdot\mathbf{i} = V_x,\ \mathbf{V}\cdot\mathbf{j} = V_y \text{ and } \mathbf{V}\cdot\mathbf{k} = V_z. \quad . \quad . \quad (2\cdot9)$$

The scalar product assumes an important form in rectangular co-ordinates. Writing from Equation 1.5,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \text{ and } \mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}.$$

Equations 2.5, 2.6 and 2.8 give

$$\mathbf{A}\cdot\mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$
$$= A_xB_x + A_yB_y + A_zB_z. \quad . \quad (2.10)$$

That is, the scalar product of two vectors is the sum of the products of their components along each of the coordinate axes. If, for example, **A** is a force and **B** is a displacement, the total work done is equal to the sum of the works done by the components of force and the corresponding components of displacement, as is physically obvious.

3. Line and Surface Integrals as Scalar Products. In Fig. 8, let OA be any curve drawn in a vector field and

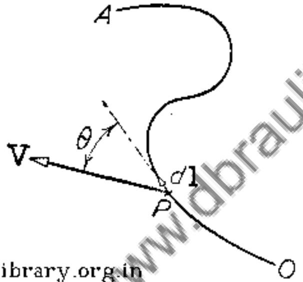


FIG. 8.—Tangential Line Integral of a Vector

$d\mathbf{l}$ an element of length along it at any point P. Let **V** denote the vector at P in a direction making an angle θ with that of the length element. Then $\mathbf{V}\cdot d\mathbf{l}$ is the product of the length of the element and the component of **V** in its direction, i.e. tangentially to the curve; thus

$$\mathbf{V}\cdot d\mathbf{l} = V\cos\theta\, dl.$$

If **V** varies in magnitude and direction from point to point along the curve, the integral

$$\int_O^A \mathbf{V}\cdot d\mathbf{l} = \int_O^A V\cos\theta\, dl \quad . \quad . \quad (2.11)$$

is defined as the *line integral* of **V** along the curve OA.

Such integrals are of very frequent occurrence. For example, if **V** is a force and $d\mathbf{l}$ an element of the path of a particle along any curve, the line integral denotes the work done in displacing the particle from O to A. Again, if **V** is the electric field strength, i.e. the force on unit charge in an electric field, then the line integral expresses the potential difference between the points O and A. Further, if **V** is the velocity at any point in a fluid and the integral be taken round a closed curve, the integral is called the *circulation* of the fluid. In an electromagnetic field the line integral of the electric force round a closed path is the electromotive force in the path; in a purely electrostatic field the e.m.f. is zero.

Consider now any element of area ds upon a surface drawn in a vector field, and let **V** be the value of the vector at the middle of the element. Draw the positive* normal of unit length **n** on the element and let θ be the angle between **n** and **V**, as in Fig. 9. Then the component of

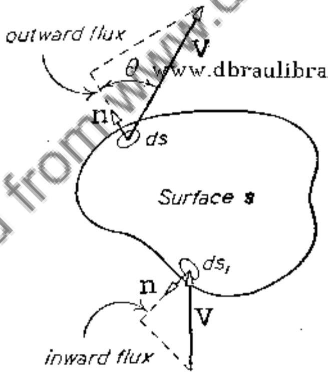


FIG. 9.—Normal Surface Integral of a Vector

V perpendicular to ds is $\mathbf{V}\cdot\mathbf{n} = V\cos\theta$ and the flux of **V** through the element is $\mathbf{V}\cdot\mathbf{n}\,ds$. The integral of this taken

* If the surface is closed, **n** is drawn outward from the enclosed volume. If the surface is unclosed, then **n** is drawn always on the same side of the surface, the normal on the opposite side being negative.

over the surface is called the *total flux* or *surface integral* of **V** through the whole surface s, i.e.

$$\iint_s \mathbf{V}\cdot\mathbf{n}\, ds = \iint_s V \cos\theta\, ds. \quad . \quad . \quad (2.12)$$

The physical meaning of such surface integrals is very simple. Suppose **V** to denote the vector velocity of a moving fluid in which a fixed surface s is drawn. At any point $\mathbf{V}\cdot\mathbf{n}\, ds$ denotes the amount of fluid passing normally through the surface element ds in unit time, and this component alone must be considered since the tangential component of **V** necessarily contributes nothing to the flow *through* the surface element. If the flow through an element is in the sense of the positive or outward normal it is counted as positive; if in the opposite sense, as on the element ds_1, it is negative. The integral of these normal contributions expresses the aggregate flow of fluid through the whole surface in unit time. If the surface is closed, positive total flux diverges from the enclosed volume while negative flux converges upon it. Should the amount of flux entering by some elements be balanced by that leaving by other elements in such a way that the total flux is zero, then either there are no sources or sinks of fluid within the enclosed volume or their sum is zero. Similar ideas apply to other fluxes, e.g. of electric or magnetic induction, of heat, &c.

4. The Vector Product. The vector product of two vectors **A** and **B** is defined as a vector having a magnitude equal to the product of the magnitudes of the factors and the sine of the angle between their directions, and a direction perpendicular to the plane containing **A** and **B**. The sense of the product along this perpendicular is defined by the right-hand screw rule; if the vector **A** is turned towards **B** through the smaller angle, the necessary rotation must be clockwise when sighting along the positive normal unit vector **n** in Fig. 10. That is, the rotation needed to move **A** to the position of **B** and the positive direction of **n** are related in the same way as the rotation and translation of a right-handed screw; this direction of **n** is that of the vector product. It follows from this definition that a change in the order of the factors in a vector product reverses the sign of the product, since $\sin(-\theta) = -\sin\theta$.

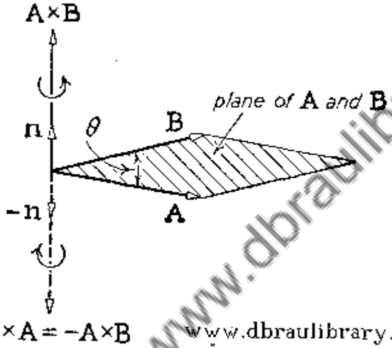


FIG. 10.—Vector Product of Two Vectors

Hence the vector product of two vectors is not commutative and the order of the terms must be strictly maintained, a further difference between vector and scalar algebras. The vector product will be denoted by a cross between the factors; then

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} = AB \sin\theta\, \mathbf{n}, \quad . \quad . \quad (2.13)$$

where **n** is a positive normal of unit length drawn from the plane containing **A** and **B**. The product vanishes not only when either factor is zero but also when the component vectors are parallel, again differing from scalar algebra. When two vectors are parallel,

$$\mathbf{A} \times \mathbf{B} = 0, \quad . \quad . \quad . \quad . \quad . \quad (2.14)$$

and when they are perpendicular,

$$\mathbf{A} \times \mathbf{B} = AB\, \mathbf{n}, \quad . \quad . \quad . \quad . \quad . \quad (2.15)$$

in which case the two vectors and their product are mutually at right angles. Applying these ideas in particular to the cartesian unit vectors **i**, **j**, **k**,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0, \quad . \quad . \quad . \quad (2.16)$$

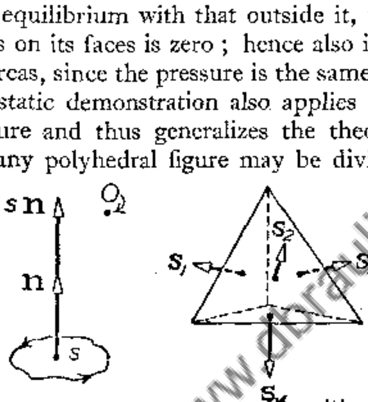
and

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}. \quad . \quad (2.17)$$

The strict cyclic order in these important results should be observed.

5. Vector Area. The magnitude of a vector product is $AB \sin \theta$, i.e. the area of the parallelogram with sides A and B and included angle θ. Since the shape of the figure is not specified by the definition of the product, any plane area of amount $AB \sin \theta$ with its positive unit normal **n** can be taken to represent a vector product. By an extension of this idea we are led to the notion of *vector area*. A plane area, such as s in Fig. 11(a), can be regarded as possessing both magnitude and direction. Its magnitude is the amount of the area and its direction is that of the normal to its plane. The sign to be attributed to a vector area is defined with reference to the order in which it is traced out as viewed from an external point such as O. If the direction of tracing is counterclockwise as seen from this viewpoint, the positive direction of the vector area is along the unit normal **n**, the directions of tracing and of **n** being related by the right-hand screw rule. Then $\mathbf{s} = s\mathbf{n}$ is the definition of a vector area **s**. Vector areas can be resolved or added just like other vector quantities.

In Fig. 11(b) a tetrahedron is shown with vectors $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, $\mathbf{s}_4$ drawn to represent the vector areas of its four triangular faces, the outward normal being regarded as positive. Resolve these areas upon any plane, e.g. one of the faces of the solid. Then some of the projections will be positive and others negative, the sum of them all being zero. Hence the total vector area of a tetrahedron is zero. This result follows at once from physical considerations, if the tetrahedron is supposed to be drawn within a fluid which is in equilibrium under hydrostatic pressure. Each face experiences a force normal to its plane and proportional to its area. Since the fluid within the solid figure is in equilibrium with that outside it, the resultant of the forces on its faces is zero; hence also is the sum of the vector areas, since the pressure is the same on all faces. This hydrostatic demonstration also applies *to any* form of solid figure and thus generalizes the theorem. Geometrically, any polyhedral figure may be divided up into



(*a*) (*b*)

FIG. 11.—(*a*) Vector Area; (*b*) Vector Area of a Closed Surface

tetrahedra; every surface introduced into the interior of the polyhedron appears twice, once with a positive and once with a negative normal. Hence for any polyhedral surface the total vector area is zero. By making the faces vanishingly small and indefinitely increasing their number we approach a closed curved surface over which

$$\iint_s d\mathbf{s} = 0.$$

6. Application to Vector Products. Consider now the vector product of a vector **A** with the sum of two others, **B** and **C**. In Fig. 12 draw a triangular prism with its parallel edges in the direction of **A** and its end faces as triangles with sides **B**, **C** and **B** + **C**. The vector areas of the triangular end faces are $\frac{1}{2}\mathbf{B} \times \mathbf{C}$ and $\frac{1}{2}\mathbf{C} \times \mathbf{B}$, which cancel; the remaining vector areas are $\mathbf{B} \times \mathbf{A}$, $\mathbf{C} \times \mathbf{A}$ and $\mathbf{A} \times (\mathbf{B} + \mathbf{C})$ in the sense of the outward normals, their sum being zero. Thus

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) + \mathbf{B} \times \mathbf{A} + \mathbf{C} \times \mathbf{A} = 0$$

i.e. $$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = -\mathbf{B}\times\mathbf{A} - \mathbf{C}\times\mathbf{A} = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C} \quad . \quad . \quad (2.18)$$

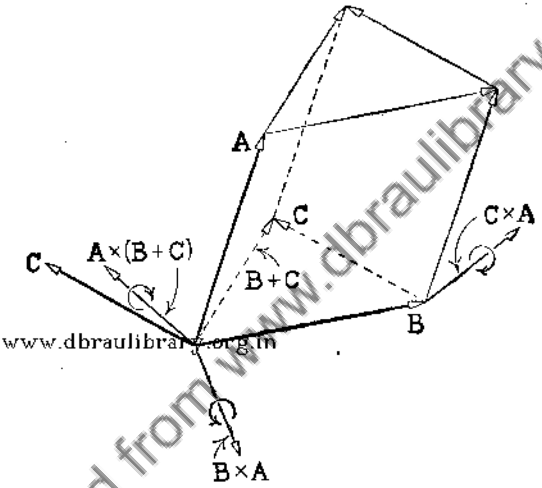


FIG. 12.—Law of Distribution for Vector Product

Hence the vector product is distributive, but the order of the factors must be strictly observed. By repeating the process it is easy to show that

$$(\mathbf{A}+\mathbf{B}+\ldots)\times(\mathbf{N}+\mathbf{O}+\ldots) = \mathbf{A}\times\mathbf{N}+\mathbf{A}\times\mathbf{O}+\ldots\mathbf{B}\times\mathbf{N}+\mathbf{B}\times\mathbf{O}+\ldots+\ldots \quad . \quad (2.19)$$

for any number of vectors.

These rules enable the vector product of two vectors to be expressed in rectangular components. Using the notation of Equation (1.5),

$$\mathbf{A}\times\mathbf{B} = (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})\times(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k})$$
$$= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \quad (2.20)$$

on applying the rules for vector products of unit vectors and the distributive law, Equations 2.16, 2.17 and 2.19. This result is more easily remembered by writing it in the form of a determinant; thus,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad . \quad . \quad . \quad (2.20\ bis)$$

7. Products of Three Vectors. The vector product of two vectors **B** and **C** being a vector, can give both a scalar and a vector product with a third vector **A**. There are, therefore, two triple products, namely **A**·(**B** × **C**) and **A** × (**B** × **C**), which occur frequently in physical applications.

The *scalar triple product* has a simple interpretation illustrated by Fig. 13. (B × C) is a vector normal to the

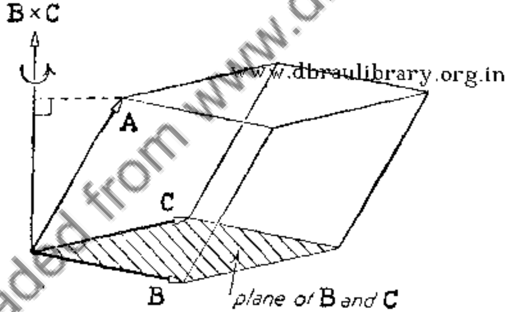


FIG. 13.—The Scalar Triple Product

plane of **B** and **C**, with a magnitude equal to the area of the shaded parallelogram. The scalar product of **A** with (**B** × **C**) is the product of this vector area and the projection of **A** along (**B** × **C**); i.e., **A**·(**B** × **C**) is the volume of the parallelepiped which has **A**, **B**, **C** for its edges. Any face of this solid can be taken as the base; hence three equivalent expressions for the volume are

$$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = \mathbf{B}\cdot(\mathbf{C} \times \mathbf{A}) = \mathbf{C}\cdot(\mathbf{A} \times \mathbf{B}), \quad . \quad (2.21)$$

cyclic order of the factors being maintained to retain the volume with positive sign.* Since the order of terms in a scalar product is immaterial, Equation 2.1, these relations are equivalent to

$$(\mathbf{A} \times \mathbf{B})\cdot\mathbf{C} = (\mathbf{B} \times \mathbf{C})\cdot\mathbf{A} = (\mathbf{C} \times \mathbf{A})\cdot\mathbf{B}\,; \quad . \quad (2.21)$$

so that the dot and cross may be interchanged at will.

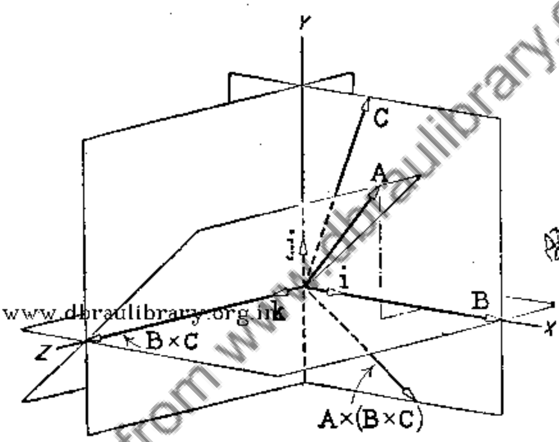


FIG. 14.—The Vector Triple Product

Three vectors, therefore, have six identical scalar triple products which may be written concisely as

$$[\mathbf{ABC}] = (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} . \quad (2.21\ bis)$$

When three vectors lie in a plane, the volume of the

* Scalar quantities of this kind, in which the sign depends on the proper cyclic arrangement of the component vector factors, are called by mathematicians 'pseudoscalars', to distinguish them from true scalars, which do not change sign when the reference axes are changed from a right-handed to a left-handed system.

parallelepiped is zero; hence the condition for vectors to be coplanar is that their scalar triple product vanishes.

The *vector triple product* $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ can also be illustrated geometrically, as in Fig. 14. The vector $(\mathbf{B} \times \mathbf{C})$ is *normal* to the plane containing **B** and **C**. Likewise, the vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is normal to the plane containing **A** and $(\mathbf{B} \times \mathbf{C})$, i.e. in the same plane as **B** and **C**. Take the X axis along **B**, the Y axis at right angles thereto in the plane of **B** and **C**, and the Z axis along $(\mathbf{B} \times \mathbf{C})$. Then in terms of their axial components the three vectors are, by Equation 1.5,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},\ \mathbf{B} = B_x\mathbf{i} \text{ and } \mathbf{C} = C_x\mathbf{i} + C_y\mathbf{j},$$

since this choice of axes makes $B_y = B_z = C_z = 0$. Expanding the vector product $(\mathbf{B} \times \mathbf{C})$ by Equation 2.20,

$$(\mathbf{B} \times \mathbf{C}) = B_xC_y\mathbf{k};$$

and also by the same rule,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = A_yB_xC_y\mathbf{i} - A_xB_xC_y\mathbf{j}.$$

But from Equation (2.10)

$$\mathbf{A}\cdot\mathbf{C} = A_xC_x + A_yC_y \text{ and } \mathbf{A}\cdot\mathbf{B} = A_xB_x,$$

so that

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= A_yC_y\mathbf{B} - A_xB_xC_y\mathbf{j} \\ &= (A_yC_y + A_xC_x)\mathbf{B} - A_xC_xB_x\mathbf{i} - A_xB_xC_y\mathbf{j} \\ &= (A_xC_x + A_yC_y)\mathbf{B} - A_xB_x(C_x\mathbf{i} + C_y\mathbf{j}).\end{aligned}$$

Substituting from above,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}. \quad . \quad . \quad (2.22)$$

Each term of the product involves the external factor **A** in a scalar product, first with the extreme and then with the middle factor. Hence the value of a vector triple product is entirely different if the order of the factors be interchanged.

Products of more than three vectors do not often occur in physical applications and when encountered they are easily reduced by use of the preceding theorems. The results serve to confirm what is already shown by Equations (2.21), that vectors do not follow the law of association for multiplication, p. 3.

8. Summary. Chapters I and II give all the essential rules for the manipulation of vector algebra. By extending the meaning of the terms addition and subtraction to the geometrical processes explained on p. 6, sums and differences of vectors follow the ordinary laws of commutation and association, as shown by Equations 1.2 and 1.3. It is, therefore, immaterial what order or manner of grouping of the vectors is followed when adding or subtracting them. Vectors and scalars are so far in agreement, but important differences appear when products are considered.

The scalar product of two vectors is commutative and distributive, Equations 2.1 and 2.7, and vanishes when either factor is zero, exactly as in ordinary algebra. It differs, however, in vanishing when the vectors are perpendicular. The vector product of two vectors differs considerably from the ordinary rules. It is not commutative (Equation 2.13), it vanishes when the vectors are parallel, and it is distributive only if the order of the factors is maintained unchanged (Equation 2.18). For more than two vectors the non-commutative nature of the vector product causes the ordinary law of association to be not obeyed, as has been demonstrated in Section 7 of this chapter.

Mathematically, vectors can be defined as quantities that require for their manipulation a non-commutative algebra which has the agreements with, and differences from, ordinary scalar algebra that have been summarized above. Vector algebra is only one of many non-commutative algebras known to mathematicians, but it is among those that have the greatest practical utility.

CHAPTER III

THE DIFFERENTIATION OF VECTORS

1. Scalar Differentiation. Let **V** be a vector which is undergoing a continuous change of magnitude and direction. In Fig. 15 $\delta\mathbf{V}$ is a small increment which gives

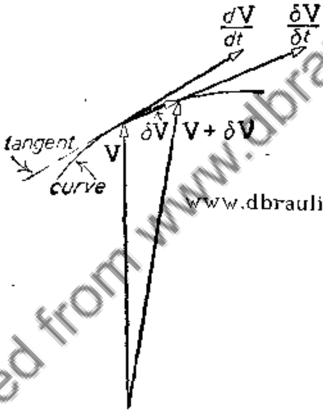


FIG. 15.—Scalar Differentiation of a Vector

the new value $\mathbf{V} + \delta\mathbf{V}$ for the vector. Suppose that **V** is a vector function of a scalar variable t; then when t changes from t to $t + \delta t$, **V** becomes $\mathbf{V} + \delta\mathbf{V}$. The ratio $\delta\mathbf{V}/\delta t$ is the average rate of change of **V** with t, and as δt becomes vanishingly small the ratio attains a limiting value which is the rate of increase of **V**, i.e.

$$\frac{d\mathbf{V}}{dt} = \text{Lim}\left[\frac{\delta\mathbf{V}}{\delta t}\right] \text{ as } \delta t \text{ becomes zero.}$$

This is the *derivative* of **V** with respect to the scalar variable t. As **V** varies, its extremity moves over a curve and the derivative is a vector in the direction of the tangent to this curve at each point. Second, third and higher derivatives are obtained by similar arguments, by analogy with the ordinary ideas of scalar calculus.

When **V** is expressed in rectangular co-ordinates,

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}\ ;$$

V_x, V_y, V_z are now functions of t. Since **i**, **j**, **k** are constant vectors, it follows that

$$\frac{d\mathbf{V}}{dt} = \frac{dV_x}{dt}\mathbf{i} + \frac{dV_y}{dt}\mathbf{j} + \frac{dV_z}{dt}\mathbf{k}, \quad . \quad . \quad . \quad (3.1)$$

and similarly for higher derivatives.

A most important physical instance is when **V** is a vector of displacement and t is time; the extremity of **V** is then the path of a moving body. The derivative $d\mathbf{V}/dt$ is the velocity along the path at any instant, and $d^2\mathbf{V}/dt^2$ is the corresponding acceleration.

2. Differentiation of Sums and Products. If **V** is the sum of two vectors **A** and **B**, both of which are functions of t, then a change of t to $t + \delta t$ makes

$$\mathbf{V} + \delta\mathbf{V} = (\mathbf{A} + \delta\mathbf{A}) + (\mathbf{B} + \delta\mathbf{B}),$$

so that

$$\delta\mathbf{V} = \delta\mathbf{A} + \delta\mathbf{B}.$$

Dividing by δt and proceeding to the limit gives

$$\frac{d\mathbf{V}}{dt} = \frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}, \quad . \quad . \quad . \quad (3.2)$$

so that the operation of differentiation is distributive, as in ordinary calculus.

For the scalar product $\mathbf{V} = \mathbf{A}\cdot\mathbf{B}$ the increment in t gives

$$\mathbf{V}+\delta\mathbf{V} = (\mathbf{A}+\delta\mathbf{A})\cdot(\mathbf{B}+\delta\mathbf{B}) = \mathbf{A}\cdot\mathbf{B}+\delta\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\delta\mathbf{B}+\delta\mathbf{A}\cdot\delta\mathbf{B},$$

expanding the scalar product by Equation 2.8. Subtracting $\mathbf{A}\cdot\mathbf{B}$, dividing by δt and proceeding to the limit, neglecting second-order terms,

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}. \quad . \quad . \quad . \quad (3.3)$$

The order of the factors can be changed at will, i.e. the operation is commutative. If $\mathbf{B} = \mathbf{A}$ we have the useful form for the self product,

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \frac{d}{dt}(\mathbf{A}^2) = \frac{d}{dt}(A^2) = 2\mathbf{A}\cdot\frac{d\mathbf{A}}{dt} = 2A\frac{dA}{dt} \quad . \quad (3.3a)$$

For the particular case of a constant vector $\mathbf{A}^2 = A^2 = constant$, defining the position of a point on the surface of a sphere of radius A, dA/dt is zero and Equation 3.3a leads to $\mathbf{A}\cdot(d\mathbf{A}/dt) = 0$. Hence $d\mathbf{A}/dt$ is perpendicular to $\mathbf{A}$. If a point moves on the surface of a sphere its velocity is always normal to the radius vector, as is physically obvious.

For the vector product $\mathbf{V} = \mathbf{A} \times \mathbf{B}$ expansion by Equation 2.19 gives

$$\mathbf{V} + \delta\mathbf{V} = (\mathbf{A} + \delta\mathbf{A}) \times (\mathbf{B} + \delta\mathbf{B}) = \mathbf{A} \times \mathbf{B} + \delta\mathbf{A} \times \mathbf{B} + \mathbf{A} \times \delta\mathbf{B} + \delta\mathbf{A} \times \delta\mathbf{B},$$

leading to

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}, \quad . \quad . \quad (3.4)$$

in which the order of the factors must be strictly maintained.

Putting $p = d/dt$ the reader can easily verify that for triple products

$$p[\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})] = p\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) + \mathbf{A}\cdot(p\mathbf{B} \times \mathbf{C}) + \mathbf{A}\cdot(\mathbf{B} \times p\mathbf{C}),$$

and

$$p[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] = p\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{A} \times (p\mathbf{B} \times \mathbf{C}) + \mathbf{A} \times (\mathbf{B} \times p\mathbf{C}).$$

In general, differentiation in vector analysis is seen to follow the same rules as in ordinary differential calculus, except so far as vector algebra differs from scalar algebra in the non-commutative property of a vector product.

3. Partial Differentiation. These simple properties of differentiation as applied to vectors can be extended to partial derivatives when a vector is a function of more than one scalar independent variable. The most useful case is that of a vector **V** which is a function of the cartesian co-ordinates x, y, z of a point in space. If y and z remain constant while x increases, the partial derivative $\partial\mathbf{V}/\partial x$ denotes the rate of increase of **V** with respect to x. Likewise changing y and z alone gives the partial derivatives $\partial\mathbf{V}/\partial y$ and $\partial\mathbf{V}/\partial z$, denoting the rates of increase with respect to y and z respectively. If now x, y and z change simultaneously by differential increments dx, dy, dz, the total change or total differential of **V** will be

$$d\mathbf{V} = \frac{\partial\mathbf{V}}{\partial x}dx + \frac{\partial\mathbf{V}}{\partial y}dy + \frac{\partial\mathbf{V}}{\partial z}dz, \quad . \quad . \quad (3{\cdot}5)$$

which is of frequent occurrence in physical applications of vector analysis. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector from the origin, then its differential increment is

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}.$$

Equation 3.5 may be symbolically written as

$$d\mathbf{V} = \left[\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right]\mathbf{V}.$$

If we now define the operator ∇ by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

it is easy to verify that the scalar product of ∇, regarded as a kind of vector, with $d\mathbf{r}$ gives the operator in square brackets. Thus,

$$d\mathbf{V} = (\nabla\cdot d\mathbf{r})\mathbf{V}. \quad . \quad . \quad . \quad . \quad (3{\cdot}5\ bis)$$

The operator ∇ is of immense importance in physical applications of vector analysis, wherein it appears in association with both scalar and vector operands. These uses will be fully explained in Chapters IV and V.

CHAPTER IV

THE OPERATOR ∇ AND ITS USES

1. The Operator ∇. The differential operator ∇ was introduced by Sir William Rowan Hamilton and developed by P. G. Tait; it is of central importance in all three-dimensional physical problems. The symbol ∇ was originally named 'nabla' after a harp-like ancient Assyrian musical instrument of similar shape; other writers have called it 'atled', i.e. 'delta' reversed. It is now usual to adopt the term 'del' introduced by J. Willard Gibbs. In cartesian notation

$$\text{del} = \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}, \quad . \quad . \quad . \quad (4.1)$$

which may be applied as a directive differentiator either to a scalar or to a vector function of space. Again, treating the differentiators in ∇ as scalars, we may formally regard ∇ as a vector which can have either a scalar or a vector product with other vectors. In vector analysis there are three fundamental operations with ∇ which are of physical interest. If S is a scalar function and $\mathbf{V}$ a vector function of space, these operations are (i) ∇S, where ∇ acts as a differentiator; (ii) $\nabla \cdot \mathbf{V}$, and (iii) $\nabla \times \mathbf{V}$, where ∇ is treated as a formal vector. (See also p. 113).

2. The Gradient of a Scalar Field. On p. 12 it has been shown that certain physical quantities, such as temperature or electric potential or any such non-directed quantity, can be represented from point to point in space by a scalar point-function, S, of the co-ordinates. The entire scalar field can be mapped out by level surfaces, upon each of which the scalar function S has a constant value. Consider two such surfaces very close together and examine a small portion of them in the neighbourhood of a given point A on the surface characterized by the constant value S of the scalar function; the second surface is specified by a constant value $S + dS$. This is shown,

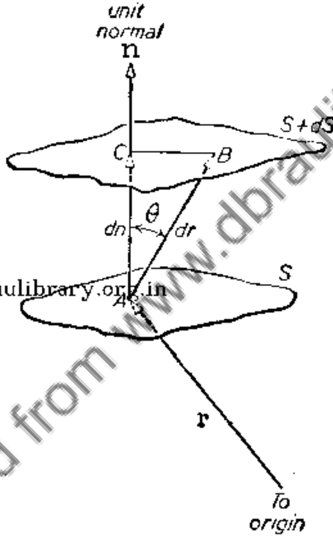


FIG. 16.—Gradient of a Scalar Point Function

much magnified, by Fig. 16. If $\mathbf{r}$ is the radius vector from the origin to the point A, any point such as B in the second surface is given by $\mathbf{r} + d\mathbf{r}$. The least distance between the surfaces will be AC, in the direction of the unit normal vector $\mathbf{n}$ at A and of length dn.

If dr be the length of AB, the magnitude of the rate of increase at A of S in the direction of AB will be $\partial S/\partial r$ when the two surfaces are vanishingly close together.

This rate of increase becomes greatest in the direction of the unit normal **n**, i.e. along AC, when it has the value $\partial S/\partial n$. Note that

$$\frac{\partial S}{\partial r} = \frac{\partial S}{\partial n}\cos\theta.$$

Hence, if **n** is the unit vector normal to the level surface at any point in a scalar field, the vector $\mathbf{n}\partial S/\partial n$ gives the greatest rate of increase of S at the point in magnitude and direction. This vector is called the *gradient* of S at the point and is written

$$\text{grad } S = \frac{\partial S}{\partial n}\mathbf{n} \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

Thus, the gradient of a scalar field is a vector field, the vector at any point having a magnitude equal to the most rapid rate of increase of S at the point and in the direction of this fastest rate of increase, i.e. perpendicular to the level surface at the point.

A simple physical example will fix the reader's ideas. Suppose S is the potential in an electric field due to static charges. Then the electric force at any point is in the direction of the greatest rate of *decrease* of potential, i.e. normal to the equipotential surfaces, and has a magnitude equal to that rate of decrease. That is, the electric force is $-$ grad S and is obviously a vector field.

It is clear from the way that the idea of gradient has been introduced that it is an intrinsic property of a scalar field and is, therefore, a physical notion entirely independent of any particular system of co-ordinate axes. In other words, the operation denoted by grad is *invariant*.

2a. The Operation ∇S. Consider now the vector represented by

$$\nabla S = \frac{\partial S}{\partial x}\mathbf{i} + \frac{\partial S}{\partial y}\mathbf{j} + \frac{\partial S}{\partial z}\mathbf{k}$$

in rectangular co-ordinates. The vector rates of increase of S in the directions of the axes of x, y and z are $\mathbf{i}\partial S/\partial x$, $\mathbf{j}\partial S/\partial y$ and $\mathbf{k}\partial S/\partial z$. Their sum will be a vector with the magnitude and direction of the total or most rapid rate of increase of S. It remains to show that this expression is equivalent to the gradient as just defined. To do this, take the scalar product on both sides of the gradient equation with an element of radius vector $d\mathbf{r}$, i.e. resolve $d\mathbf{r}$ in the direction of the normal at A in Fig. 16. Then

$$(\text{grad } S)\cdot d\mathbf{r} = \frac{\partial S}{\partial n}\mathbf{n}\cdot d\mathbf{r} = \frac{\partial S}{\partial n}dr \cos\theta = \frac{\partial S}{\partial n}dn = dS,$$

since $\partial S/\partial n$ is the total normal rate of change of S. Now in rectangular co-ordinates,

$$dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz\,;$$

hence,

$$(\text{grad } S)\cdot d\mathbf{r} = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz$$

$$= \left(\frac{\partial S}{\partial x}\mathbf{i} + \frac{\partial S}{\partial y}\mathbf{j} + \frac{\partial S}{\partial z}\mathbf{k}\right)\cdot(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) = (\nabla S)\cdot d\mathbf{r}\,;$$

so that

$$\text{grad } S = \nabla S = \frac{\partial S}{\partial x}\mathbf{i} + \frac{\partial S}{\partial y}\mathbf{j} + \frac{\partial S}{\partial z}\mathbf{k}, \quad (4{\cdot}3)$$

showing that the operations grad and ∇ applied to a scalar point-function are identical. Also since the gradient in a scalar field is independent of the nature of co-ordinates, so also is the operator ∇; it is a mere convenience, therefore, to express it in terms of x, y and z. We may regard ∇ as a *directional differentiator* which, when applied to a scalar function of space, gives the greatest rate of change of the function in magnitude and direction at every point, i.e. derives from a scalar field its vector field of gradient.

We have here an important case in which a vector field is derived from a scalar field by the process of finding the gradient of the latter. It does not necessarily follow conversely that *all* vector fields can be expressed as the gradient of a scalar function. Let $\mathbf{V}_S$ be a vector which is derived from a scalar S in the form

$$\mathbf{V}_S = \text{grad } S = \nabla S.$$

In Fig. 17 draw any path, such as that marked Path 1,

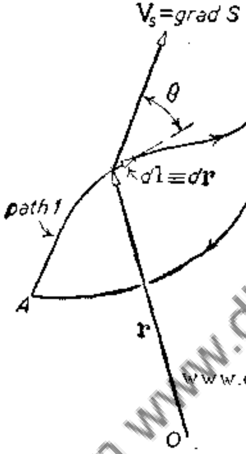


Fig. 17.—Line Integrals in a Lamellar Field

between two points A and B in the vector field and let $\mathbf{V}_S$ make an angle θ with the element $d\mathbf{l}$ of the path; then the product of the length of the element and the component of $\mathbf{V}_S$ in its direction is, from p. 18,

$$V_S \cos\theta \, dl = \mathbf{V}_S \cdot d\mathbf{l}.$$

The path is traced out by the extremity of a radius vector $\mathbf{r}$ from the origin; then $d\mathbf{l}$ is equivalent to $d\mathbf{r}$ and we may write, from p. 36,

$$\mathbf{V}_S \cdot d\mathbf{l} = (\nabla S) \cdot d\mathbf{r} \equiv (\text{grad } S) \cdot d\mathbf{r} = dS.$$

Hence the line integral of $\mathbf{V}_S$ from A to B is

$$\int_A^B \mathbf{V}_S \cdot d\mathbf{l} = \int_A^B (\text{grad } S) \cdot d\mathbf{r} = \int_A^B dS = S_B - S_A, \quad . \quad (4.4)$$

where S_A and S_B are the values of the scalar at the extremities of the path. Since only the end values are concerned, the line integral has the same value independently of the path followed.

Now take a closed path, consisting of Path 1 from A to B followed by Path 2 from B to A. From Equation (4.4), since the limits of integration are reversed for the second path, the value of the line integral along it is $S_A - S_B$; hence round a closed path

$$\oint (\text{grad } S)\cdot d\mathbf{l} = 0 \quad . \quad . \quad . \quad . \quad (4.5)$$

Summarizing, *when a vector field can be expressed as the gradient of a scalar field*, the line integral of the vector taken between two points is independent of the path followed and is equal to the difference between the values of the scalar at its ends; further, the line integral round any closed path in such a vector field is zero.

A vector field $\mathbf{V}_S$ derived from a scalar S by the relation $\mathbf{V}_S = \text{grad } S$ is sometimes called a *scalar potential field*, S being the potential of $\mathbf{V}_S$. Since space is divided up into layers or laminae by the level or equipotential surfaces of the function S, $\mathbf{V}_S$ is also called a *lamellar vector*. Again, since the essential property of such a vector is that its line integral round any closed path is zero, the vector field is called a *non-curl field*, a term which will be explained in Section 4.

A simple example is the electrostatic field of charged conductors, which can be specified by a system of equipotential surfaces. The vector field of electric force is derived from the scalar potential as its negative gradient, i.e. the electric force is in the direction of the greatest rate of fall of potential and has a magnitude equal to that rate. The line integral of the electric force between two points is the potential difference between them and is independent of the path taken; it represents the work done in moving a unit charge of positive electricity from one point to the other. Round a closed path the work done is zero, expressed by the statement that the electromotive force vanishes round any closed path in a static electric field. The reader will be able to construct other physical examples of lamellar vector fields, e.g. in the flow of heat with temperature isothermals or in gravitational attraction with level surfaces.

3. The Divergence of a Vector Field. In Fig. 18

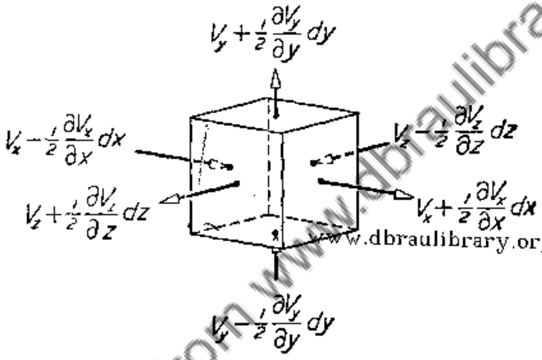


FIG. 18.—Divergence of a Vector Point Function

let **V** be the value of a vector function at the middle of an infinitesimal element of volume with sides dx, dy and dz parallel to the axes of x, y and z. The vector **V** has axial components of magnitude V_x, V_y and V_z. To fix ideas the reader may think of **V** as a vector giving the velocity of a moving fluid in magnitude and direction.

Consider the two faces of the volume element, each with area $dy\,dz$ perpendicular to the axis of X. On the left-hand face the value of the x component of the vector at the middle of the face becomes

$$V_x - \tfrac{1}{2}\frac{\partial V_x}{\partial x}dx,$$

and may be taken as the value all over the face when this becomes vanishingly small. Similarly, on the right-hand face the x component is

$$V_x + \tfrac{1}{2}\frac{\partial V_x}{\partial x}dx.$$

We now define the flux through any face as the scalar product of the vector area of the face and the vector $\mathbf{V}$; i.e. as the product of the area of the face and the normal component of the vector upon it, see p. 19. The flux is positive when the component of $\mathbf{V}$ and the outward-drawn normal on the face of the element are in the same sense. Then the excess of flux leaving the element over that entering it in the X direction is

$$\left(V_x+\tfrac{1}{2}\frac{\partial V_x}{\partial x}dx\right)dy\,dz-\left(V_x-\tfrac{1}{2}\frac{\partial V_x}{\partial x}dx\right)dy\,dz=\frac{\partial V_x}{\partial x}dx\,dy\,dz.$$

In the hydromechanical case this represents the net volume of fluid passing per second in the X direction. By similar reasoning the contributions parallel to Y and Z are

$$\frac{\partial V_y}{\partial y}dx\,dy\,dz \quad \text{and} \quad \frac{\partial V_z}{dz}dx\,dy\,dz.$$

The total net flux diverging from or leaving the element is, by Equation 2.12,

$$\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)dx\,dy\,dz.$$

The amount of this flux per unit volume is defined as the *divergence* of the vector $\mathbf{V}$ and is written

$$\text{div}\,\mathbf{V}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z} \quad . \quad . \quad . \quad (4.6)$$

Since the divergence is the *amount* of flux, it is essentially scalar.

If the divergence exists at a point in a fluid, whether liquid or gas, and is positive, it expresses the rate at which fluid is flowing away from the point per unit volume thereat.

Hence, either the fluid is expanding and its density at the point is falling with time, or the point is a *source* at which fluid is entering the field. When the divergence is negative it gives the rate at which fluid is flowing toward the point per unit volume. In this case either the fluid is contracting and its density rising at the point, or the point is a negative source, a *sink*, at which fluid is leaving the field. Since most practical liquids are almost incompressible, the existence of divergence in them means the presence of a volume distribution of sources or sinks rather than changes of density. In the case of non-material fluxes, such as those of the thermal, electric or magnetic fields, the existence of divergence means the presence of a source or sink of flux at the point. For example, in the electric field, positive divergence means that there is positive electricity at the point; in the thermal field the point is either a source of heat or a place where the temperature is falling.

When the divergence is everywhere zero, the flux entering any element of space is exactly balanced by that leaving it and we may write,

$$\text{div } \mathbf{V} = 0, \quad . \quad . \quad . \quad . \quad . \quad (4.7)$$

which is true in many practical problems. In a fluid this means that there can be no sources or sinks in the field, nor can its density be changing; i.e. the fluid is incompressible. If the fluxes entering and leaving an element are equal, none can have been generated within it; the lines of flow of the vector **V** must either form closed curves (cf. the magnetic field of a current), or terminate upon bounding surfaces (cf. the electric field in a condenser), or extend to infinity. A vector which satisfies this condition is called *solenoidal* (from a Greek word meaning a tube).

3a. The Operation ∇·V. Consider now the scalar product of the operator ∇ and the vector **V**. Expressed in rectangular co-ordinates the rules for scalar products, Equations 2.5, 2.6 and 2.8 give

$$\nabla\cdot\mathbf{V}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(V_x\mathbf{i}+V_y\mathbf{j}+V_z\mathbf{k})=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z},$$

that is,

$$\nabla\cdot\mathbf{V}\equiv\text{div }\mathbf{V} \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

Now the operation ∇ has been shown on physical grounds to be invariant, i.e. independent of any system of axes. Hence the idea of divergence is also invariant. This is, indeed, physically obvious, since the amount of flux entering or leaving per unit volume at a point is clearly a conception quite independent of any system of co-ordinates that may be used to express the shape and position of the volume element.

4. The Curl of a Vector Field. It has been shown on p. 38 that when a vector field can be derived as the gradient of a scalar field, the line integral of the vector taken round any closed path is zero; this result is true no matter what size or shape the path may have. A vector field satisfying this condition is known as a lamellar field, and is of a special, though very important, class. Many vector fields occur in physical problems, however, in which the closed-path line integral is not zero and which cannot, therefore, be expressed as the gradient of a scalar point function; it is to an important property of these more general fields that we now give attention.

Consider a very small region of such a vector field, several lines of flow in which are shown by Fig. 19(*a*); the portion is chosen small enough for the lines to be regarded as nearly straight and parallel. Into this field put a small plane area, shown for convenience as a rectangle. When the area is perpendicular to the field, as in position 1, none of the field is directed along any part of the bounding edge of the area; the line integral round it is zero. In position 2, with the area parallel to the field, since the value of the vector along the upper edge is assumed to be different from that along the lower edge, the line integral round the boundary has a finite value. Similar arguments apply to intermediate positions; the value of the line integral depends, therefore, upon the direction of the normal to the area relative to the field,

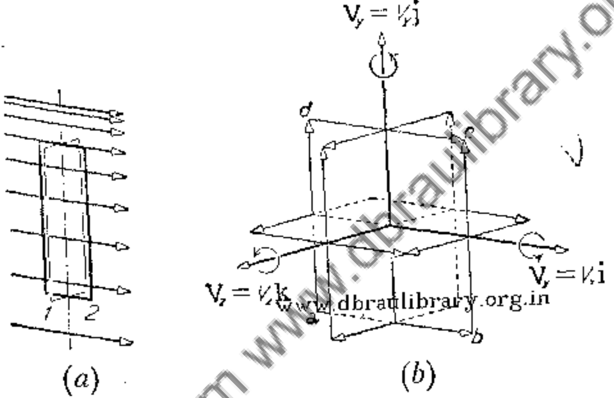


FIG. 19.—Curl of a Vector Point Function

i.e. upon the orientation of the given small vector area at the region considered.

In general, if we put a small vector area of any shape at any point in a vector field and compute the line integral of the vector **V** around its bounding edge there will be an orientation of the area for which the line integral is greatest. The amount of this maximum line integral expressed per unit area is called the *curl* of the vector field at the point, and is given the vectorial sense of the positive normal drawn on the small exploring area when in the position giving this greatest integral. Some writers use the term *rotation* (briefly *rot.*) to denote the same conception, since the curl is associated in hydromechanics with the rotation or spin of a fluid in which the particles have angular velocity. Physical examples illustrating the meaning of the curl of a vector field will be given later; the meaning depends very much upon the nature of the problem.

To calculate the curl in terms of its cartesian components, take three infinitesimal rectangular areas intersecting mutually at right angles at a point where the vector field **V** has components of magnitude V_x, V_y, V_z as in Fig. 19(*b*). Taking the positive normals to the areas along the positive directions of the X, Y and Z axes respectively, the circular arrows indicate the positive senses in which their boundaries must be traversed to accord with the right-hand screw rule for vector areas.

As an example, take one of these areas such as *abcd* in Fig. 19(*b*) with sides *dx*, *dy*, its normal being along the axis of Z. Since the rectangle is very small, the numerical value of the component of **V** at the middle of any side may reasonably be taken as the average value along that side; the arrows show the directions in which the components act. Since V_x, V_y, V_z are functions of the co-ordinates (x, y, z) of the middle of the rectangle, the average values along the four sides *ab*, *bc*, *dc*, *ad* will be respectively

$$V_x - \tfrac{1}{2}\frac{\partial V_x}{\partial y}dy,\ V_y + \tfrac{1}{2}\frac{\partial V_y}{\partial x}dx,\ V_x + \tfrac{1}{2}\frac{\partial V_x}{\partial y}dy \text{ and } V_y - \tfrac{1}{2}\frac{\partial V_y}{\partial x}dx.$$

Around the contour *abcd* the line integral is, therefore,

$$\left[\left(V_x - \tfrac{1}{2}\frac{\partial V_x}{\partial y}dy\right) - \left(V_x + \tfrac{1}{2}\frac{\partial V_x}{\partial y}dy\right)\right]dx + \left[\left(V_y + \tfrac{1}{2}\frac{\partial V_y}{\partial x}dx\right) - \left(V_y - \tfrac{1}{2}\frac{\partial V_y}{\partial x}dx\right)\right]dy,$$

that is,

$$\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy.$$

Since the area of the element is *dx dy*, the bracketed term is the magnitude of the component curl of the vector field taken about the Z axis. Giving it the sense of unit vector **k**, therefore, we may write

$$\text{curl}_Z \mathbf{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k}.$$

By following a precisely similar method with the two remaining rectangles we find the component curls about the Y and X axes to be

$$\text{curl}_Y \mathbf{V} = \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{j} \text{ and } \text{curl}_X \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{i}.$$

Adding the three components gives

$$\text{curl } \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k} \quad (4.9)$$

which can be conveniently written in the form of a determinant as

$$\text{curl } \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} \quad . \quad . \quad (4.9 \ bis)$$

4a. The Operation $\nabla \times \mathbf{V}$. Take the vector product $\nabla \times \mathbf{V}$ of the operator ∇ and the vector **V**, expressing the result in rectangular co-ordinates. Using Equation 2.20 for expanding a vector product,

$$\nabla \times \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k})$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k};$$

that is,

$$\nabla \times \mathbf{V} = \text{curl } \mathbf{V} \quad . \quad . \quad . \quad (4.10)$$

The operation is, of course, independent of the system of axes.

5. Simple Examples of Curl. The meaning of the operation curl may now be illustrated by a few simple examples. First consider a rigid body which is rotating with an angular velocity ω about an axis OA, O being a fixed point in the body. Then any point P, Fig. 20,

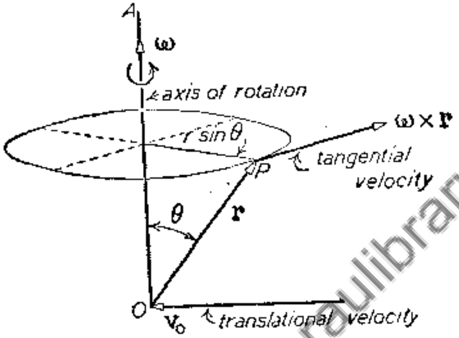


FIG. 20.—Curl and Angular Velocity

moves in a circular path about OA with a tangential linear velocity $\omega r \sin\theta$, where r is the distance from O to P.

In vector notation the angular velocity is a vector $\boldsymbol{\omega}$ drawn along the axis in a sense related to the rotation by the right-hand screw rule; the co-ordinate of P is the radius vector $\mathbf{r}$. Then the tangential velocity at P is $\boldsymbol{\omega} \times \mathbf{r}$, which has magnitude $\omega r \sin\theta$ and is perpendicular to the plane containing $\boldsymbol{\omega}$ and $\mathbf{r}$.* If in addition the

* To verify that it is physically correct to treat an angular velocity as a vector it is only needful to show that two such velocities compound as vectors, see p. 7. Let $\boldsymbol{\omega}_1$ be the angular velocity about an axis through O; then the tangential velocity at P is $\boldsymbol{\omega}_1 \times \mathbf{r}$. Now let $\boldsymbol{\omega}_2$ be the angular velocity about a second axis through O inclined to the first; then P has a tangential velocity due to this rotation of $\boldsymbol{\omega}_2 \times \mathbf{r}$. But linear velocities compound vectorially, hence the total linear velocity at P is

$$\boldsymbol{\omega}_1 \times \mathbf{r} + \boldsymbol{\omega}_2 \times \mathbf{r} = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \mathbf{r} \equiv \boldsymbol{\omega} \times \mathbf{r},$$

by the distributive law, Equation 2.18. Hence the motion is the same as that due to an angular velocity equal to the vector sum of the two components. A similar theorem is obviously true of infinitesimal angular rotations and of angular accelerations; it is not true of finite angular rotations.

whole body is moving with a linear velocity $\mathbf{v}_0$ in any direction, the total velocity at P is

$$\mathbf{V} = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r},$$

and $$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \nabla \times \mathbf{v}_0 + \nabla \times \boldsymbol{\omega} \times \mathbf{r}.$$

Now since $\mathbf{v}_0$ is constant for all points in the body and is thus independent of x, y, z, $\nabla \times \mathbf{v}_0$ is clearly zero.

The angular velocity $\boldsymbol{\omega}$ is also a constant vector for all points and can be written as

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k},$$

where its components ω_x, ω_y, ω_z are independent, therefore, of the coordinate $\mathbf{r}$ of the point P. Writing

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and using Equation 2.20 makes

$$\boldsymbol{\omega} \times \mathbf{r} = (\omega_y z - \omega_z y)\mathbf{i} + (\omega_z x - \omega_x z)\mathbf{j} + (\omega_x y - \omega_y x)\mathbf{k}.$$

Again, using Equation 2.20 with Equation 4.1 and remembering that since ω_x, ω_y, ω_z are not functions x, y, z their derivatives are zero, it is easy to verify that

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = 2\omega_x\mathbf{i} + 2\omega_y\mathbf{j} + 2\omega_z\mathbf{k} = 2\boldsymbol{\omega}.$$

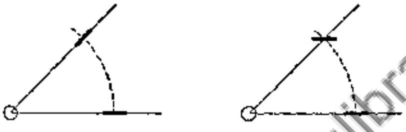
Finally, therefore,

$$\text{curl } \mathbf{V} = 2\boldsymbol{\omega}$$

and $$\boldsymbol{\omega} = \tfrac{1}{2} \text{ curl } \mathbf{V} \quad . \quad . \quad . \quad . \quad . \quad (4.11)$$

Hence when a rigid body is in motion the curl of its linear velocity at any point gives twice its angular velocity in magnitude and direction. (See also p. 113.)

Now examine the motion of a fluid and consider what may happen to an infinitesimal volume within it. The volume can have three kinds of motion simultaneously: (i) It may be moving with a linear velocity of translation as a whole. (ii) If the fluid is deformable the element may change its shape. (iii) It may be in rotation. At any instant the little volume may be regarded as a rigid body; the curl of the velocity of the fluid at the point where the element is situated gives twice its angular velocity, the instantaneous axis of rotation being that of the curl. The nature of the rotational motion will be made clearer by Fig. 21(*a*), which shows two positions of



(*a*) (*b*)

FIG. 21.—Rotational (*a*) and Irrotational (*b*) Motion

a small portion of the fluid in movement about an axis at *O*. It is clear that the portion has rotated; and if every elementary portion of the volume immediately round *O* has rotated by the same amount, curl **V** would give twice the angular velocity of rotation about *O*. By contrast, Fig. 21(*b*) shows a small portion of fluid which, in its motion about *O*, does not rotate; hence there is no curl **V** and its angular velocity is zero.

If a motion is such that the velocity has a curl, the ultimate particles of the body are in rotation with an instantaneous angular velocity. The motion is described as *rotational* or *vortical*. If, on the other hand, the particles do not rotate, there is no curl and the motion is *irrotational* or *non-vortical*.

Another example of an entirely different kind occurs in the magnetic field of a conductor carrying a steady current. At any point in the field put a very small plane area and turn it into such a position that the line integral of the magnetic force taken round its boundary is the greatest possible; this value expressed per unit area is the vector **curl H**, i.e. the magneto-motive force per unit area at the point. If the point is within the material of the conductor at a place where the vector of current density is **c**, then this will be the total current passing normally per unit area through the elementary path when the line integral round it is greatest. In electromagnetic units, curl $\mathbf{H} = 4\pi\mathbf{c}$, so that the curl of the magnetic force at any point is proportional to the current density thereat and has the same direction. For a point in the field external to the conductor there is no current density and curl $\mathbf{H} = 0$.

6. Divergence of a Vector Product. In certain physical problems, e.g. in calculating the flux of energy in an electromagnetic field, it is required to find the vector product of two vectors and then to work out the divergence of the result. Treating this at present as an exercise in manipulation of vector operations, we are to find div $(\mathbf{A} \times \mathbf{B})$. Using Equation 2.20, write

$$\begin{aligned}\mathbf{V} = \mathbf{A}\times\mathbf{B} &= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \\ &= V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}.\end{aligned}$$

Now apply Equation 4.6 and

$$\begin{aligned}\text{div } \mathbf{V} &= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \\ &= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &\quad - A_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - A_y\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) - A_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right).\end{aligned}$$

Using Equations 2.10 and 4.9 gives

$$\text{div } (\mathbf{A} \times \mathbf{B}) = \mathbf{B}\cdot \text{curl } \mathbf{A} - \mathbf{A}\cdot \text{curl } \mathbf{B} \quad . \quad . \quad (4.12)$$

7. Divergence and Curl of $S\mathbf{A}$. In later work we require the divergence and curl of a vector field which is itself the product of a scalar field S and a vector field $\mathbf{A}$ at every point. The components of the product are SA_x, SA_y and SA_z.

Using Equation 4.6 to find the divergence,

$$\begin{aligned}\text{div } S\mathbf{A} &= \frac{\partial}{\partial x}(SA_x) + \frac{\partial}{\partial y}(SA_y) + \frac{\partial}{\partial z}(SA_z) \\ &= S\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + \left(A_x\frac{\partial S}{\partial x} + A_y\frac{\partial S}{\partial y} + A_z\frac{\partial S}{\partial z}\right) \\ &= S \text{ div } \mathbf{A} + (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot\left(\frac{\partial S}{\partial x}\mathbf{i} + \frac{\partial S}{\partial y}\mathbf{j} + \frac{\partial S}{\partial z}\mathbf{k}\right),\end{aligned}$$

which from Equation 4.3 is

$$\operatorname{div} S\mathbf{A} = S \operatorname{div} \mathbf{A} + \mathbf{A} \cdot \operatorname{grad} S \quad . \quad . \quad . \quad (4.13)$$

Using Equation 4.9 *bis* the curl can be written

$$\operatorname{curl} S\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ SA_x & SA_y & SA_z \end{vmatrix},$$

the x component of which has the magnitude

$$\frac{\partial}{\partial y}(SA_z) - \frac{\partial}{\partial z}(SA_y) = S\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \left[A_z\frac{\partial S}{\partial y} - A_y\frac{\partial S}{\partial z}\right].$$

Using Equation 4.9, the first bracket is the x component of curl **A**; the second bracket is the x component of (grad S) × **A**, as may be verified from Equations 4.3 and 2.20. Working out the other components gives finally

$$\begin{aligned} \operatorname{curl} S\mathbf{A} &= S \operatorname{curl} \mathbf{A} + (\operatorname{grad} S) \times \mathbf{A} \\ &\equiv S \operatorname{curl} \mathbf{A} - \mathbf{A} \times \operatorname{grad} S \quad . \quad . \quad (4.14) \end{aligned}$$

CHAPTER V

FURTHER APPLICATIONS OF THE OPERATOR ∇

1. The Operator div grad. In Equations 4.2 and 4.3 it has been shown that if S is a scalar function of position in space, then grad $S \equiv \nabla S$ is a vector function having the magnitude and direction of the greatest rate of increase of S; this vector is the gradient of S and the flux lines of grad S cut normally through the level surfaces of S. Since grad S is a vector it can have a divergence; using Equations 4.8 and 2.4 this scalar quantity is

$$\text{div grad } S = \nabla\cdot(\nabla S) = \nabla\cdot\nabla(S) = \nabla^2 S.$$

From Equations 4.1 and 4.3,

$$\text{div grad } S = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial S}{\partial x}\mathbf{i} + \frac{\partial S}{\partial y}\mathbf{j} + \frac{\partial S}{\partial z}\mathbf{k}\right).$$

Expanding by the usual rules for scalar products, Equations 2.5, 2.6 and 2.8,

$$\text{div grad } S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}, \quad . \quad . \quad . \quad (5.1)$$

and the operation

$$\text{div grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2, \quad . \quad . \quad (5.2)$$

which is known as *Laplace's operator*. Examples of its use will be found in later chapters.

2. The Operator curl grad. Since grad S is a vector, it is possible to calculate its curl and thus interpret the operation curl grad applied to a scalar point-function. Now from Equation 4.9 *bis* and above, since the components of the vector for which the curl is to be found are $\partial S/\partial x$, $\partial S/\partial y$ and $\partial S/\partial z$,

$$\begin{aligned}\text{curl grad } S \\ = \nabla \times (\nabla S) =\end{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial S/\partial x & \partial S/\partial y & \partial S/\partial z \end{vmatrix} = \text{zero} \quad . \quad (5.3)$$

This result follows at once from physical considerations. On p. 38 it has been shown that $\mathbf{V}_S$ = grad S is a lamellar vector, of which S is the scalar potential. The characteristic feature of a lamellar field is that the line integral of $\mathbf{V}_S$ round *any* closed path is zero. Since the curl of a vector field is a particular kind of closed-path line integral, curl $\mathbf{V}_S$ = curl grad S is necessarily zero as $\mathbf{V}_S$ is lamellar. A typical non-curl field of this class is the electric force due to static electric charges.

3. The Operator ∇^2 with Vector Operand. A vector point-function $\mathbf{V}$ may be expressed in terms of its rectangular components $V_x\mathbf{i}$, $V_y\mathbf{j}$, $V_z\mathbf{k}$ in the usual way.

Since V_x, V_y and V_z are each scalar functions of position (x, y, z), Equation 5.2 applies to them all. Then if

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k},$$

$$\nabla^2\mathbf{V} = \nabla^2 V_x\mathbf{i} + \nabla^2 V_y\mathbf{j} + \nabla^2 V_z\mathbf{k}, \quad . \quad . \quad (5.4)$$

which is a result of great importance in electromagnetic theory and hydromechanics.

4. The Operator grad div. If $\mathbf{V}$ is a vector field, div $\mathbf{V}$ is a scalar field, which, therefore, has a gradient. This new vector, of which div $\mathbf{V}$ is the potential, is necessarily lamellar because curl grad on scalar operand is zero; it may be written as

$$\text{grad div } \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$= \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y \partial z}\right)\mathbf{j}$$

$$+ \left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2}\right)\mathbf{k} \quad . \quad (5.5)$$

CHAPTER VI

THE THEOREMS OF GAUSS AND STOKES

1. The Divergence Theorem of Gauss. Consider a closed surface s drawn in a vector field $\mathbf{V}$; the normal flux through an element of vector area $d\mathbf{s} = \mathbf{n}ds$, $\mathbf{n}$ being the outward drawn normal, is $\mathbf{V}\cdot d\mathbf{s} = \mathbf{V}\cdot\mathbf{n}ds$, as explained on p. 19. The surface integral of such elements will give the total normal flux through the surface in the form given by Equation 2.12,

$$\iint_s \mathbf{V}\cdot d\mathbf{s} = \iint_s \mathbf{V}\cdot\mathbf{n}ds.$$

Referring to Fig. 24, an elementary volume dv within

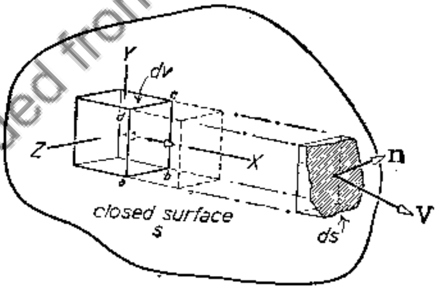


FIG. 24.—Gauss's Theorem

the surface s is shown; a small cube has been taken for convenience, but any shape will suffice. The size of the element is much exaggerated, merely for ease in drawing the diagram. The total flux diverging from this volume is div $\mathbf{V}dv$, $\mathbf{V}$ being the vector at the middle of the element, and this flux has been computed on p. 40 by calculating the surface integral of the normal components of $\mathbf{V}$ through the surfaces bounding the volume. For the face *abcd* the positive direction of the x component of $\mathbf{V}$ and the outward-drawn normal are in the same sense and the flux is positive. A contiguous cube, shown dotted, has the same component of $\mathbf{V}$ acting through the common face—since $\mathbf{V}$ and its components are assumed to be continuous, as also are the derivatives of these quantities—but its outward normal, shown dotted, is in the opposite sense and the flux is negative. The surface fluxes through the common face cancel. This argument can be applied by adding further elements along the X axis until we reach an element which has one face in the surface s, shown shaded; this alone makes a contribution to the normal flux through s. Applying the same treatment to the surfaces of all elements of volume throughout the space enclosed by s, we arrive at a total flux $\iint_s \mathbf{V}\cdot d\mathbf{s}$. But at the same time we have integrated div $\mathbf{V}dv$ throughout the enclosed volume and this also measures the total flux. Equating them,

$$\iint_s \mathbf{V}\cdot d\mathbf{s} = \iint_s \mathbf{V}\cdot\mathbf{n}\,ds = \iiint \text{div } \mathbf{V}dv = \iiint \nabla\cdot\mathbf{V}dv, \quad (6.1)$$

which is *Gauss's theorem of divergence.*

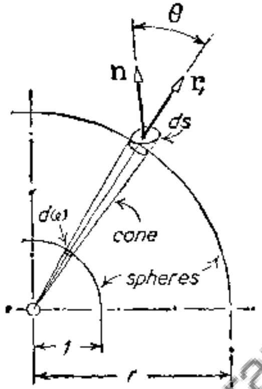
Although the deduction of this theorem is not a rigid mathematical proof—many such can be found in the larger text-books—it is based on self-evident physical facts. Consider, for example, a closed surface drawn within a fluid which is moving at a given point with vector velocity $\mathbf{V}$. The total amount of fluid passing per second through the surface can be found in two equivalent ways. First, by calculating $\mathbf{V}\cdot d\mathbf{s}$, i.e. the product of an element of surface and the component of velocity perpendicular to it, for every element of the surface and adding all the contributions. Second, by investigating the divergence of a volume element, i.e.

the excess of sources of fluid over the sinks contained in it per unit volume, and integrating div $\mathbf{V}dv$ through the enclosed volume. The two results are physically equivalent, since the excess of fluid leaving the surface over that entering it must be due to the net amount of fluid injected into the portion of the field within s by the aggregate of sources and sinks.

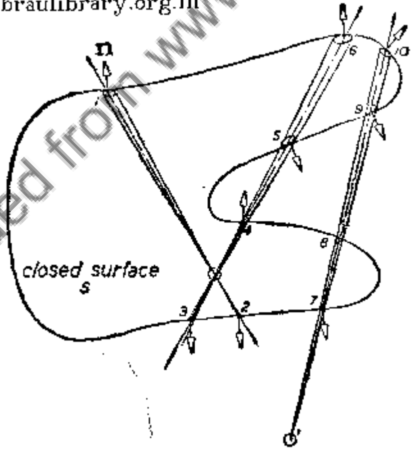
2. Gauss's Theorem and the Inverse Square Law. In Fig. 25(a) let ds be an elementary portion of a surface, $\mathbf{n}$ being unit positive normal upon it. From a point O draw a conical pencil touching the boundary of ds and let $\mathbf{r}_1$ be unit vector in the direction of the radius vector $\mathbf{r} = r\mathbf{r}_1$ from O to the element. Draw spheres of radii r and unity respectively. If $d\omega$ is the area cut by the cone from the surface of the sphere of unit radius, $d\omega/1^2 = ds \cos\theta/r^2$; $d\omega$ is called the solid angle subtended by ds at O. The solid angle is regarded as positive when the angle θ is acute and negative when obtuse. The total solid angle subtended by a spherical surface is 4π.

Fig. 25(b) shows any closed non-spherical surface. When O is inside the surface an element at 1, 2, or 3 contributes a positive solid angle. A cone joining O to elements 4, 5, and 6 shows that these subtend equal solid angles at O, but while the contributions of 4 and 6 are positive that of 5 is negative. In general, therefore, any small cone from O cuts the surface an odd number of times, equivalent to a single contribution of $d\omega$. Hence the entire surface s subtends at O a total solid angle of 4π. When the vertex of the cone is at O' outside the surface, the surface is cut an even number of times with alternately negative and positive equal contributions, as shown at 7, 8, 9, 10. Hence the surface s subtends a solid angle of zero at an external point. These two facts are a purely geometric consequence of the definition of solid angle.

Consider now the Newtonian potential field of a point



(a)



(b)

Fig. 25.—Gauss's Theorem and the Inverse Square Law

unit mass or other similar physical entity placed either at O or at O'. Taking its potential as $1/r$, Equation 5.11 shows that we can put in Equation 6.1

$$\mathbf{V} = \text{grad}\,(1/r) = -\mathbf{r}_1/r^2$$

and $$\mathbf{V}\cdot d\mathbf{s} = -(\mathbf{r}_1/r^2)\cdot\mathbf{n}ds = -\cos\theta\, ds/r^2 = -d\omega,$$

noting that $\mathbf{r}_1$ and $\mathbf{n}$ are unit vectors with an angle θ between them. Then

$$\iint_s \mathbf{V}.d\mathbf{s} = -\iint d\omega \quad . \quad . \quad . \quad (6.2)$$

Hence the total closed surface integral of the vector grad $(1/r)$, e.g. of the total flux of force from a point unit mass, is -4π when the mass is inside the surface and zero when outside it. The minus sign means that the flux is inward, see also Fig. 23. In both cases the flux is independent of the position of the mass, which may be anywhere inside or outside the surface. This form of Gauss's theorem is extremely useful and will be used in Chapter VII.

3. Stokes's Theorem. In a vector field $\mathbf{V}$ draw any unclosed surface or cap having for its bounding edge a given closed curve, and calculate the line integral of $\mathbf{V}$ round the closed curve when it is traced out counter-clockwise as shown by the boundary arrows in Fig. 26. Then from Equation 2.11 the value of this integral will be

$$\oint \mathbf{V}\cdot d\mathbf{l},$$

the circle indicating that a closed path has been traversed.

By means of a system of intersecting lines drawn in the unclosed surface let it be divided up into infinitesimal surface elements. Consider the shaded element of vector area $\mathbf{n}ds$, $\mathbf{n}$ being the positive unit normal upon it. The boundary of the element is traced out counter-clockwise, this sense and the direction of **n** following the usual right-hand screw rule, see p. 22. At the middle of ds the vector field will have a curl and the scalar product of this with the vector area will give the component of the line integral of **V** round the boundary of ds, i.e. **n**·curl **V**ds, as explained on p. 43. A similar process can be applied to all surface elements, tracing them all out in the same way. Since **V** and its derivatives, and hence curl **V**, are

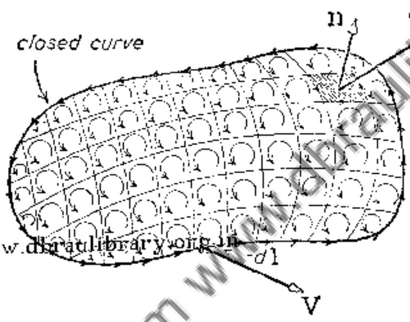


Fig. 26.—Stokes's Theorem

all supposed to be finite and continuous functions, the line integral along the common sides of contiguous elements will cancel, leaving only a contribution from sides that lie in the given closed curve. Adding the contributions of all elements yields the closed line integral, but in obtaining it a surface integral of **n**·curl **V**ds has been simultaneously made ; hence,

$$\oint \mathbf{V}\cdot d\mathbf{l} = \iint \mathbf{n}\cdot\text{curl } \mathbf{V}ds. \quad . \quad . \quad . \quad (6.3)$$

This is *Stokes's theorem* and states that the tangential line integral of a vector function round any closed curve is equal to the normal surface integral of the curl of that function over any unclosed surface which has the curve for its bounding edge; rigid mathematical proofs taking account of all the necessary continuity conditions would be out of place here and will be found in the usual text-books. The converse theorem is necessarily true, i.e. if the line integral of **V** round a closed curve is equal to the surface integral of **A** over any open surface bounded by the curve, then **A** = curl **V**.

When curl **V** is zero, Equation 6.3 shows that the closed-path line integral of **V** vanishes; this has been shown on p. 38 to be a characteristic feature of a lamellar field.

4. Invariance of Divergence and Curl. Gauss's and Stokes's theorems may be regarded from several points of view. Their physical meaning has already been stressed, the first theorem providing alternative ways of expressing the flux of a vector through a closed surface, the second giving equivalent ideas for the line integral of a vector round a closed curve. Both results are necessarily scalar quantities. From another standpoint we may look upon them as useful analytical transformations which enable certain integrals to be more simply expressed. Equation 6.1 shows that the volume integral of a vector function can be expressed as a surface integral over a closed surface confining the volume. Equation 6.3 reduces the integral of a vector over an open surface to one taken round its boundary edge.

Returning to Equation 6.1, confine the volume integral to the space within an infinitesimal element dv and the surface integral to its enclosing surface, then

$$\iint \mathbf{V}\cdot d\mathbf{s} = \text{div } \mathbf{V}\, dv.$$

The divergence at a point is the value of this surface integral per unit volume as the element is made vanishingly small, i.e.

$$\text{div } \mathbf{V} = \text{Lim.} \left[\frac{1}{dv} \iint \mathbf{V}\cdot d\mathbf{s} \right] \text{ as } dv \text{ becomes zero.}$$

Similarly, apply Stokes's theorem to an infinitesimal plane area at a given point, turning the area until the line integral of **V** round its boundary has the largest value; then **n** and curl **V** are in the same direction and their scalar product is the magnitude of the curl, say *curl* **V**, giving

$$\oint_{max.} \mathbf{V}\cdot d\mathbf{l} = curl\ \mathbf{V}\ ds.$$

Hence, as ds is made vanishingly small,

$$curl\ \mathbf{V} = \text{Lim.}\left[\frac{1}{ds}\oint_{max.} \mathbf{V}\cdot d\mathbf{l}\right]$$

and has the direction of **n** in the position of maximum line integral.

The ideas of divergence and curl have been shown on physical grounds to be conceptions independent of any system of co-ordinates in which they may be expressed; they are *invariant.* The formulae derived above can be regarded as a purely mathematical justification of this property, since no co-ordinates are involved in their deduction. Some mathematical treatises, indeed, treat these formulae as definitions of div and curl, building up the theory of vector analysis thereon.

CHAPTER VII

THE SCALAR POTENTIAL FIELD

1. General Properties. Let S be a scalar point-function which may be mapped out in space by a series of *level surfaces*, upon each of which the scalar has a definite but different constant value. These surfaces divide up the region of space into a series of layers or laminae. Associated therewith is a vector field $\mathbf{V}_S$ directed everywhere normal to the level surfaces, i.e. in the direction of the greatest rate of increase of S at any point and having a magnitude equal to that rate of increase. This is expressed by Equation 4.3,

$$\mathbf{V}_S = \text{grad } S = \nabla S.$$

Fig. 27 illustrates these conditions for small portions of

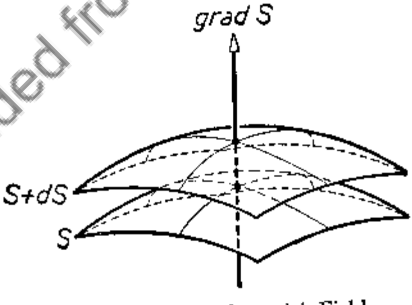


FIG. 27.—The Scalar Potential Field

two infinitesimally-close level surfaces, characterized by values of S and $S + dS$ respectively.

Referring again to Fig. 17 and Equation 4.4, it has been shown that such a vector field has the essential property that the value of the tangential line integral of $\mathbf{V}_S$ along *any* path joining two points A and B is the same and equal to the difference between the values of S at B and A, i.e.

$$\int_A^B \mathbf{V}_S \cdot d\mathbf{l} = S_B - S_A.$$

Hence round any closed path the line integral of $\mathbf{V}_S$ is zero; in particular round a path of infinitesimal size the integral also vanishes, i.e. curl $\mathbf{V}_S$ is zero.

Because of the relation between $\mathbf{V}_S$ and S, $\mathbf{V}_S$ is called a *lamellar* or *scalar potential vector field*, S being its *potential*. Since the curl is zero, the field is also called a *non-curl* or *irrotational* field.

2. The Inverse Square Law. Point Sources. In practice the greatest interest is found in lamellar fields based upon the law of inverse squares. Consider a point source of vector flux, around which the level surfaces are concentric spheres; the flux lines are radial straight lines diverging from the source. At unit distance from the source the spherical surface has an area of 4π; if the source is of unit strength this is indicated by drawing one flux line per unit of area to represent the magnitude of the vector field at unit distance. A total vector flux of 4π lines cuts normally through all spherical surfaces and hence the magnitude of the vector field at any point varies with the inverse square of the distance from the source, since the spherical area varies directly as the square of the distance. If the *strength* of the source is q, the flux is $4\pi q$.

In some fields the point is an actual source of material flux, e.g. when liquid is entering at a point within a fluid; the vector is then the outward radial velocity of the fluid. In most cases, however, the flux is non-material, e.g. of heat, or of gravitational or electric forces. Note that in the case of gravity the point mass can only attract another mass placed in its field; the force is essentially radially inward. But in hydromechanics, heat, electricity, and magnetism both positive and negative sources, the latter called *sinks*, can exist, making possible both outward and inward fluxes. Fluid or heat may either enter or leave; electric or magnetic forces may be either repulsive or attractive. Masses act in the vectorial sense in the same way as sinks.

It will be convenient to fix ideas by examining an actual example, namely, the field of electric point charges *in vacuo*. Two such charges of positive electricity have been shown by Coulomb's experiments to be repelled with a force kq_1q_2/r^2 in the line joining them. If unit charge is defined as repelling a similar charge 1 cm. distant with a

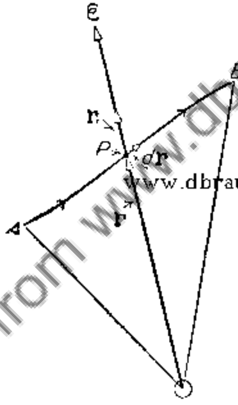


FIG. 28.—Electric Field of Point Charge

force of 1 dyne, we have the ordinary electrostatic system of units, $k = 1$ and q_1q_2/r^2 is the force in dynes between charges q_1 and q_2 c.s.u. Let a positive charge q be put at O, Fig. 28, then the force per unit positive charge placed at P is called the electric force or field strength; its magnitude is q/r^2 and its direction that of the unit radius vector $\mathbf{r}_1$, i.e. of $\mathbf{r}$, then

$$\mathfrak{E} = \frac{q}{r^2}\mathbf{r}_1.$$

The potential difference between two points A and B is defined as the work done *against* the forces of the field in moving the unit positive charge from A to B by any path. Since like charges repel, positive work must be done against the field forces to bring the charges nearer together. The work that would be done *by* the field forces is $\mathbf{\mathcal{E}}\cdot d\mathbf{r}$; hence the potential difference is, by definition,

$$-\int_A^B \mathbf{\mathcal{E}}\cdot d\mathbf{r} = -q\int_A^B \frac{1}{r^2}\mathbf{r}_1\cdot d\mathbf{r} = -q\int_A^B \frac{dr}{r^2} = q\left[\frac{1}{OB} - \frac{1}{OA}\right].$$

Now let A be at infinity—where $\mathbf{\mathcal{E}}$ is zero—and make B coincide with P; then the integral becomes

$$-\int_\infty^P \mathbf{\mathcal{E}}\cdot d\mathbf{r} = \frac{q}{r}.$$

This is the *potential* at P, namely, the work done on unit positive charge in bringing it from infinity to the point P by any path. Using Equation 5.11 shows that

$$\mathbf{\mathcal{E}} = \frac{q}{r^2}\mathbf{r}_1 = -\operatorname{grad}\left(\frac{q}{r}\right).$$

A similar argument can be adapted to other fields. In general, let F be a flux vector at a distance r from a source of strength q; then the Newtonian potential is defined as

$$\phi = q/r \; ; \qquad (7.1)$$

and

$$\mathbf{F} = -\operatorname{grad}\phi = -\operatorname{grad}(q/r), \qquad (7.2)$$

of which the three cartesian components are respectively

$$F_x = -\partial\phi/\partial x,\ F_y = -\partial\phi/\partial y \text{ and } F_z = -\partial\phi/\partial z \quad . \qquad (7.3)$$

The vector F points in the direction of the greatest rate of *decrease* of ϕ. For example, if ϕ is temperature it is clear that the flow of heat will be in the direction in which the temperature falls most rapidly.

Since potentials are scalar functions of space they are arithmetically additive. Thus, if there are several point sources $q_1, q_2 \ldots$ at distances $r_1, r_2 \ldots$ from a point in the field, the potential is

$$\phi = \left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \ldots\right] \quad . \quad . \quad . \quad (7.4)$$

This may be extended still further. If the sources are continuously distributed with density σ per unit area over boundary surfaces the summation becomes the integral

$$\phi = \iint \frac{\sigma ds}{r} \quad . \quad . \quad . \quad . \quad . \quad (7.5)$$

taken over all the surfaces. Each surface element is regarded as a point source yielding a Newtonian potential.

In Equation 5.12 it has been shown that except at a point occupied by a source the field **F** is solenoidal, i.e. div **F** = 0 ; then

$$\nabla^2\phi = 0 ; \quad . \quad . \quad . \quad . \quad . \quad (7.6)$$

i.e. Laplace's equation is satisfied by the potential at all source-free points in space, since div grad ϕ is, by Equation 5.12, zero. A field of this kind is, therefore, known as a *Laplacian field* and ϕ is called a harmonic function. Equations 7.1, 7.4 and 7.5 may be regarded as solutions of Equation 7.6 for various physical conditions.

3. Volume Distributions. In many problems the field is due to volume distributions of sources, each element of which acts according to the inverse square law. Familiar examples are the electric field due to a space charge of electrons or of ionized gas molecules, the attraction of solid bodies, the flow of internally generated heat (as in an electrically-heated conductor), &c. Let ρ be the density of sources per unit volume within a closed surface s ; ρ will be a scalar function of position. Then if dv is a volume element in Fig. 29, ρdv is an elemental source. Two cases arise and will be separately examined.

Let P be outside the surface s, then the total potential at P is

$$\phi = \iiint \frac{\varrho dv}{r} \quad . \quad . \quad . \quad . \quad . \quad (7.7)$$

taken over the volume enclosed by s. Also, since ϱdv is constant for each differentiating operation,

$$\mathbf{F} = -\text{grad}\ \phi \text{ and } \nabla^2\phi = 0,$$

exactly as before. The field external to a volume distribution is solenoidal, i.e. the potential is Laplacian.

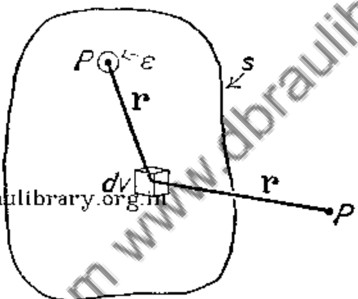


FIG. 29.—Potential of a Volume Distribution

If P is inside the surface s there is a small difficulty, since r in the denominator of the integrand can now become zero. Surround P by a small spherical surface of radius ε; then, taking the volume integral through the space confined between s and the sphere, P is outside this region and the integral is finite. Now let the sphere be made vanishingly small; then in the limit, its volume approaches dv, which varies with ε^3. Since r varies with ε, dv/r depends on ε^2 and vanishes as ε becomes zero. Hence Equation 7.7 expresses also the potential inside the surface if it is regarded as the limiting value reached when a small sphere round P is reduced without limit; the field is not, however, Laplacian. In Equation 6.2 it has been shown that when $\mathbf{V} = \text{grad}\ (1/r)$ the total normal flux of $\mathbf{V}$

through a surface enclosing a source of unit strength is -4π independently of the position of the source. Hence when $\mathbf{F} = -\text{grad}\,\phi$, where ϕ is due to sources of total amount q, no matter how distributed within the surface s, the flux is $+4\pi q$. In the present case q is the volume integral of the source-density, so that

$$\iint_s \mathbf{n}\cdot\mathbf{F}ds = 4\pi\iiint \varrho dv.$$

Applying the divergence theorem, Equation 6.1,

$$\iint_s \mathbf{n}\cdot\mathbf{F}ds = \iiint \text{div}\,\mathbf{F}dv = 4\pi\iiint \varrho dv,$$

so that $\qquad \text{div}\,\mathbf{F} = 4\pi\varrho = -\text{div grad}\,\phi$

and $\qquad \nabla^2\phi = -4\pi\varrho, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.8)$

which is *Poisson's equation.* Hence the field in a region where there is a volume distribution of sources is not solenoidal. There is divergence, i.e. total normal flux per unit volume, at any point, of an amount equal to 4π times the density of sources per unit volume.

It is obvious that the same results will hold for two or more distributed sources, interpreting the integral, Equation 7.7, as extending over all the enclosed volumes. In the space between them the field is solenoidal and the potential Laplacian; within each volume the field has a Poissonian potential and has divergence.

4. The Potential Operation. The integral, Equation 7.7, taken over all space may be considered to be the solution of Poisson's differential equation, Equation 7.8, for any distribution of sources throughout space. The operation of finding the potential due to a given distribution of source density ϱ is indicated by the notation pot ϱ, due to Gibbs. Then, making use of Equations 7.7 and 7.8,

$$\phi = \iiint \frac{\varrho dv}{r} = \text{pot}\,\varrho = -\frac{1}{4\pi}\,\text{pot}\,\nabla^2\phi \quad . \quad (7.9)$$

Hence the operations

$$-\frac{1}{4\pi}\text{pot}(\quad)\text{ and }\nabla^2(\quad) \quad . \quad . \quad (7.9a)$$

acting on a scalar operand are inverse, one undoing the effect of the other in much the same way that integration and differentiation are inverse operations. Applications of this notation have been much developed by Gibbs, and a few simple extensions of the idea will be found in the next Chapter.

5. Multivalued Potentials. In mathematical text-books it is shown that the operations grad, div, and curl,

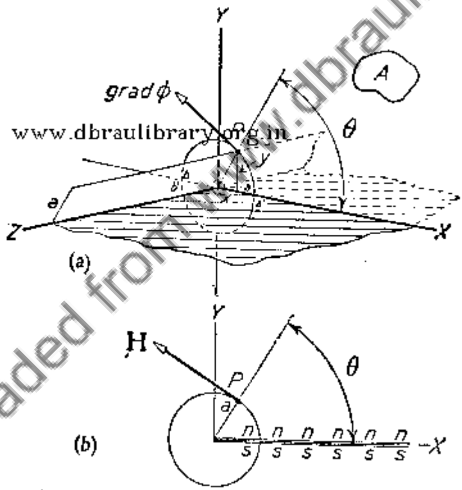


FIG. 30.—Multivalued Scalar Potential

as well as theorems such as those of Gauss and Stokes, are valid provided that the scalar or vector point functions to which they are applied remain finite, free from discontinuities and single valued within any given region of space and that the same conditions also restrict their derivatives in any direction up to the second order (so that operations such as ∇^2 may have a meaning). Fortunately, the conditions of finiteness and continuity are well observed in most fields of physics and engineering, but important cases do arise in electromagnetism and hydromechanics where the idea of scalar potential leads to multivalued functions.

Consider first a simple mathematical example, the potential given by

$$\phi = \arctan (y/x) = \theta \pm k\pi,$$

where k is any positive integer including zero. Then the 'equipotential' surfaces are radial planes θ = constant intersecting along the Z axis, as shown in Fig. 30(a). This intersection violates the condition of single-valuedness mentioned on p. 12; the axis is termed a *singularity*. Taking the gradient,

$$\text{grad}\ \phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\arctan\frac{y}{x}\right)$$

$$= -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} = \frac{1}{a}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$$

The lines of the vector, grad ϕ, are in the direction of the counter-clockwise tangent at any point P, perpendicular to the 'equipotential' plane through that point. To calculate the line integral of grad ϕ we need

$$\nabla\phi\cdot d\mathbf{a} = \left[-\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}\right]\cdot[dx\mathbf{i} + dy\mathbf{j}] = \frac{xdy - ydx}{x^2+y^2}$$

Consider a circular path round O in the plane of XY; then, as a is constant, from

$$x = a\cos\theta \text{ and } y = a\sin\theta$$

we have

$$dx = -a\sin\theta d\theta,\ dy = a\cos\theta d\theta \text{ and } xdy - ydx = a^2d\theta\,;$$

hence

$$\int\nabla\phi\cdot d\mathbf{a} = \int d\theta$$

along any portion of a circular path. Since it is independent of a the integral applies to *any* path.* For a closed path such as A, $\int d\theta$ is clearly zero, since to trace it out the angle subtended at O by the area is passed over once positively and once negatively by the radius vector from O. This is true even when A is infinitesimal and hence curl grad ϕ is zero, as required by Equation 5.3.

Now trace a semicircular path ab, giving a line integral of π. Likewise, a line integral along the semicircle $a'b'$ gives $-\pi$, i.e. the same magnitude but of opposite sign. But the beginnings and ends of the two paths are the same; hence the value of the line integral depends on which path is followed as well as upon its end points. Round any closed path linked about the axis of Z the line integral is $\pm 2\pi$ per tour, the upper sign relating to rotation in the sense from X to Y. Such a closed path cannot be reduced to zero without cutting through the Z axis and cannot satisfy Equation 5.3. The potential at P, defined as on p. 70, is multi-valued, being increased or diminished by 2π for every tour made round the axis of Z before arriving at P from an infinite distance.

Mathematicians describe a region of space in which these phenomena occur as 'cyclic' about a singularity in the region; in our problem the singularity is the axis of Z where all the level surfaces intersect. A path such as A which can be contracted to zero without enclosing or cutting through the singular point or points is 'reducible'; a path such as $abb'a'$ is 'irreducible' since it cannot be made zero, but can only be made to fit more

* Any path can be resolved into three components, viz. parallel to OZ, radially from OZ and around it. The first two can add nothing to the line integral, since the lines of flow of grad ϕ are circular. At any radius a the path around the axis is $ad\theta$, for a small angular rotation. The magnitude of grad ϕ being $1/a$, the element of line integral is $d\theta$.

and more closely round the singularity. A cyclic region can be made acyclic by inserting impassable barriers to prevent the drawing of irreducible paths, thereby isolating the singularity. For example, in Fig. 30(a) take the semi-infinite plane of ZX to the right of the Z axis as a barrier, shown shaded. Then θ cannot now exceed 2π and ϕ is thereby made single valued and the conditions of potential theory are satisfied.

The problem just examined is easily seen to represent the magnetic field of a current i electromagnetic units in a thin straight wire extending to infinity along the Z axis if we take

$$\phi = -2i \arctan (y/x).$$

Then the magnetic force is

$$\mathbf{H} = -\operatorname{grad} \phi = (2i/a)(-\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j})$$

and the line integral of $\mathbf{H}$ round a closed path encircling the current is $4\pi i$, the so-called magnetomotive force. Remove the current, and let the half-plane of ZX to the right be occupied by a thin uniformly-magnetized sheet of moment i per unit area with the north-polar surface uppermost (Fig. 30(b)). Then the magnetic field of the sheet and the current are identical in distribution, but that of the sheet is acyclic since no path can pass through it to encircle the wire. Its scalar potential is $\phi = 2i(\pi - \theta)$, θ being limited between 0 and 2π (p. 90).

The fundamental axiom of potential theory is that the line integral of the vector field is independent of the path traversed between two points ; the theory is not applicable, therefore, without some artificial aid to problems which do not satisfy this condition. The most important case is the *essentially* cyclic magnetic field of an electric current flowing in a linear circuit of any shape. Exploring the field, in the usual way, with a unit magnetic pole which is moved about from point to point, will result in changes in the amount of flux from the pole which links the circuit, inducing electromotive forces therein ; if the current is to remain constant, this will require adjustments to be made to the voltage of the battery in the circuit. For every complete tour made by the pole round the current, potential energy $\pm 4\pi i$ must be credited to the pole; this energy is, of course, derived from the battery. Physically, therefore, the scalar potential of the field *must* be multivalued, and its cyclic character has this physical explanation. To satisfy mathematical requirements, however, we replace the field of the current by that of a thin, normally-magnetized sheet with the circuit as its boundary edge—the *equivalent magnetic shell*. The potential due to the shell is $i\omega$, ω being the solid angle subtended by the circuit at any point in the field, and is single-valued as required by potential theory, p. 12; the field is then acyclic and the problem has been reduced to one in magnetostatics; see further, p. 86. It is, however, not in agreement with the essential physical nature of a current field, and must be regarded as a mere mathematical convenience. Agreement can be secured by treating the field of a linear current as a degenerate case of the vector potential field of a current-carrying medium, as on p. 84.

CHAPTER VIII

THE VECTOR POTENTIAL FIELD

1. The Magnetic Field of a Steady Current. Vector potential fields occur most commonly in electromagnetism and in hydromechanics. It will be convenient to introduce the subject by working out the properties of the magnetic field of a steady current as a practical illustration of the fundamental ideas; the reader will readily adapt them to any analogous physical problem in which he is interested.

We have seen on p. 77 that the space round a circuit carrying a current is a cyclic region; the physical meaning of this is that the work done on a unit magnetic pole in transporting it once round a constant current of i electromagnetic units is $\pm 4\pi i$ ergs. This is the well-known *circuital theorem*.

Consider now a medium, such as the interior of a copper conductor, which is carrying a current distributed in any given way. The current density at any point is $\mathbf{c}$ electromagnetic units per sq. cm. of a surface perpendicular to the direction of flow. Draw *any* closed curve in the medium and take *any* open surface having the curve for its bounding edge. If $\mathbf{n}\,ds$ is an element of vector area on the surface the current through the element is the scalar product $\mathbf{c}\cdot\mathbf{n}\,ds$, and that through the whole surface is the integral of this taken over the entire area. Now calculate the tangential line integral of the magnetic force $\mathbf{H}$ round the closed curve, in the sense of a right-hand screw relative to the unit surface-normal **n**, as in Fig. 31; then the circuital theorem makes

$$\oint \mathbf{H}\cdot d\mathbf{l} = 4\pi \iint \mathbf{c}\cdot\mathbf{n}\, ds.$$

Apply Stokes's theorem, Equation 6.3, to the left side, then

$$\iint \mathbf{n}\cdot \text{curl } \mathbf{H}\, ds = 4\pi \iint \mathbf{c}\cdot\mathbf{n}\, ds,$$

and for this to be true of *any* surface we must have

$$\text{curl } \mathbf{H} = 4\pi\mathbf{c} \quad . \quad . \quad . \quad . \quad (8.1)$$

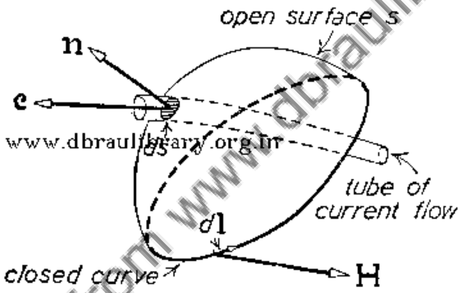


Fig. 31.—Curl of Magnetic Field due to Electric Current

Physically this means that we put a very small plane area at a point in the medium and turn it about until the line integral of **H** round its boundary is the greatest possible; the amount of the integral per unit area is the magnitude of curl **H** and the positive normal points in the direction of the curl. But the equation relates this to the current density **c**, which must, therefore, be passing normally through the small area when it is in the orientation giving the greatest value of boundary line integral.

It is to be noted that since div curl **H** is zero, Equation 5.6, div **c** is zero also. Hence currents are solenoidal, i.e. flow in closed paths. Further, since there can be no such thing as a space distribution of free magnetic poles, div **H** is zero also; the lines of magnetic force must, therefore, form closed curves.

2. The Vector Potential. On p. 55 we have seen that when a vector has curl and no divergence it may be represented as the curl of another vector, called the *vector potential*. The reason for this name will now be explained. First take at any point in the current-carrying medium,

$$\mathbf{H} = \text{curl } \mathbf{A}, \quad . \quad . \quad . \quad . \quad . \quad (8.2)$$

where **A** is the vector potential of **H**. Using Equation 5.7 on the above,

$$\text{curl } \mathbf{H} = \text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2\mathbf{A} = 4\pi\mathbf{c}.$$

We may, without any loss of generality, assume that **A**, like its related vectors **H** and **c**, is solenoidal; then div **A** is zero and

$$\nabla^2\mathbf{A} = -4\pi\mathbf{c}, \quad . \quad . \quad . \quad . \quad (8.3)$$

which is a vector form of Poisson's equation relating **A** and **c** within the medium. Using Equation 5.4, put

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \text{ and } \mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k},$$

then

$$\nabla^2 A_x = -4\pi c_x, \ \nabla^2 A_y = -4\pi c_y \text{ and } \nabla^2 A_z = -4\pi c_z.$$

If r is the distance from an element of volume, where the current density is **c**, to any point in the medium at which the components of **A** are to be found, analogy with Equations 7.8 and 7.9 gives

$$A_x = \iiint \frac{c_x\,dv}{r}, \ A_y = \iiint \frac{c_y\,dv}{r} \text{ and } A_z = \iiint \frac{c_z\,dv}{r}. \quad (8.4)$$

Recombining the components,

$$\mathbf{A} = \iiint \frac{\mathbf{c}\,dv}{r} = \text{pot } \mathbf{c} = -\frac{1}{4\pi} \text{ pot } \nabla^2\mathbf{A}\,; \quad . \quad (8.5)$$

the integrals extend throughout the volume in which **c** is distributed.

Equations 8.4 show that the components of vector potential are calculated from the components of **c** in exactly the same way as the scalar potential ϕ is found from a scalar source distribution of density ρ. Hence the name vector potential. Interpreting Equation 8.5, at the point where the vector potential is required draw a vector parallel to the current density **c** at any other point in the medium, the length of the vector being proportional to the magnitude of **c** multiplied by dv and divided by r, the distance between the points. Repeat for every element of volume into which the region containing **c** can be divided and sum the vectors obtained; i.e. integrate their effect at the point for the whole volume. The resultant vector is the vector potential at the point; the same result is more practically obtained by calculating A_x, A_y, A_z and vectorially adding these rectangular components.

By exact analogy with the scalar potential field, p. 72, the vector potential at a point outside the medium containing **c** is found by appropriate evaluation of the integrals Equations 8.4 and 8.5, but now

$$\text{curl } \mathbf{H} = \text{curl curl } \mathbf{A} = \nabla^2 \mathbf{A} \equiv 0$$

at the point. Each component of **A** satisfies Laplace's equation. **H** is now a lamellar vector and can be expressed as the negative gradient of a multivalued scalar potential Ω say. Hence at points outside a current-carrying medium the magnetic force is given by

$$\mathbf{H} = -\text{grad } \Omega = \text{curl } \mathbf{A}.$$

At the surface of separation which divides the current-carrying medium from outside space, the two expressions for **A**, in the curl and non-curl regions respectively, reduce to equality. Their analytical forms in the two media are, however, quite different.*

* A variety of practical cases will be found in the author's treatise, *Electromagnetic Problems in Electrical Engineering* (Oxford University Press, 1929).

3. The Potential Operation. The quantities **A, H, c** are all solenoidal fluxes and are related one to the other by the process of curling, Equations 8.1 and 8.2, and the calculation of potential, Equation 8.5. Summarizing these connexions,

$$\mathbf{A} = \text{pot } \mathbf{c}$$
$$\mathbf{H} = \text{curl } \mathbf{A} = \text{curl pot } \mathbf{c}$$

$$\mathbf{c} = \frac{1}{4\pi} \text{curl } \mathbf{H} = \frac{1}{4\pi} \text{curl curl } \mathbf{A} = \frac{1}{4\pi} \text{curl curl pot } \mathbf{c}.$$

The fortuitous $1/4\pi$ is a consequence of the system of units that has been used ; it may be abolished by changing to the Heaviside rationalized system. The third relation shows that the operation denoted by curl curl/4π exactly annuls the operation of computing the vector potential. Again, the first and third shows that

$$\mathbf{A} = \text{pot } \mathbf{c} = \frac{1}{4\pi} \text{pot curl curl } \mathbf{A},$$

so that pot/4π and curl curl also annul. Hence it follows that pot and curl curl are *commutative* operations ; we may, therefore, find the vector potential either before or after the operation of double curling.

Although the three vectors considered are the only ones with physical meaning, we can extend the process in either direction. Adding a fourth, **D** = curl **c**, it follows from the construction of the table that **H** = pot **D**. Any vector in the continuous series is the curl of the one immediately above it (neglecting the 4π factor) and the potential of the vector two below it. Now

$$\mathbf{H} = \text{curl pot } \mathbf{c} = \text{pot } \mathbf{D} = \text{pot curl } \mathbf{c} \; ; \quad (8.6)$$

hence it is immaterial whether we calculate the vector potential before or after calculating the curl.

These relations and many others have been developed by Gibbs into a complete calculus for manipulating solenoidal vectors, but to pursue it further would be beyond our present object.

4. Linear Currents. In electrical theory there is considerable practical interest in the magnetic field of currents flowing in infinitesimally thin conductors forming closed circuits of any configuration. Such circuits are called *linear*, since the current is caused to flow along straight or curved lines, and are a close approximation to practical circuits of thin wire. In hydromechanics, such a linear curve constitutes a vortex filament in an incompressible fluid, the fluid being in rotation about the line as axis. In both cases the field is cyclic but non-curl. The electromagnetic case is often treated magnetostatically by the 'magnetic shell' conception, but we shall now see how vector analysis enables us to express the field in terms of vector potential for a medium free from current everywhere, except in the filament constituting the linear circuit; precisely similar arguments may be applied to vortex filaments in a fluid or to any other linear singularity.

Let i be the current in the circuit and σ its vanishingly small cross-sectional area. If $\mathbf{t}$ be unit tangent at any point of the circuit, we may put $\mathbf{c} = i\mathbf{t}/\sigma$ and $dv = \sigma\, dl$, where dl is an element of length of the circuit as shown in Fig. 32; then,

$$\mathbf{c}dv = i\mathbf{t}dl = i.d\mathbf{l}$$

and

$$\mathbf{A} = i\oint\frac{d\mathbf{l}}{r} \qquad (8.7)$$

will give the vector potential at a point P when the integral is evaluated round the closed circuit. The magnetic force at P is

$$\mathbf{H} = \text{curl}\,\mathbf{A} = i\,\text{curl}\oint\frac{d\mathbf{l}}{r} = i\,\text{curl}\oint\frac{\mathbf{t}}{r}dl = i\oint\text{curl}\,\frac{\mathbf{t}}{r}\,dl,$$

since by Equation 8.6 the operations curl and pot are commutative. Using Equation 4.14, substitute $1/r$ and

t therein; remembering that the curl is calculated at P and that **t** is not a function of that point, so that curl **t** thereat is zero, gives

$$\text{curl}\,(\mathbf{t}/r) = [\text{grad}\,(1/r)] \times \mathbf{t} = -(\mathbf{r}_1 \times \mathbf{t})/r^2$$

from Equation 5.11, $\mathbf{r}_1$ being unit vector drawn from the circuit element towards P. Substituting gives

$$\mathbf{H} = -i\oint\frac{(\mathbf{r}_1 \times \mathbf{t})\,dl}{r^2} = -i\oint\frac{\mathbf{u}\sin\theta\,dl}{r^2}, \quad (8.8)$$

where **u** is a unit vector in the sense of the vector product

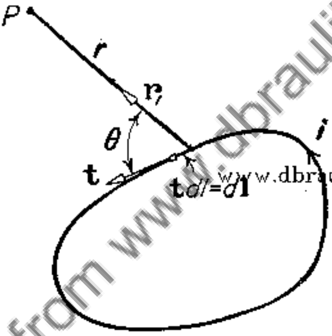


FIG. 32.—Ampère's Rule for Magnetic Force of Circuit Element (Biot-Savart Law)

of $\mathbf{r}_1$ and **t**, i.e. $\mathbf{r}_1$, **t** and **u** form a right-handed system. For example, if $\mathbf{r}_1$ and **t** are in the plane of the paper **u** is perpendicularly outward toward the reader. Equation 8.8 is known either as *Ampère's rule* or as the *Biot and Savart law*,* which states that the magnetic force due to an element dl of a closed circuit has a magnitude $i\sin\theta\,dl/r^2$ and is normal to the plane containing dl and r; the directions of magnetic force, dl and r, form a left-handed system, since **H** is in the sense of $-\mathbf{u}$.

Calculating curl **H**, since Equation 5.3 shows curl grad

* The general rule is due to Ampère for any shape of circuit. Biot and Savart's proof applies only to straight conductors.

$(1/r)$ to be zero, the field of linear currents has no curl. As shown on p. 78, scalar potential theory can be applied to the cyclic non-curl field by replacing the current by a uniformly-magnetized shell having the circuit for its bounding edge. The north polar face of the shell is directed toward a point P, as in Fig. 33, when the current as viewed

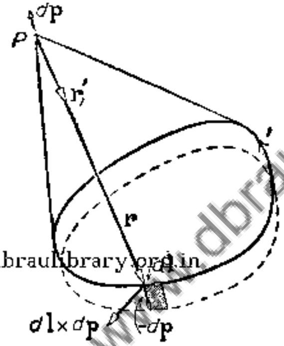


Fig. 33.—Equivalence of Magnetic Shell and Linear Current

from P appears to run counter-clockwise. If ω is the solid angle subtended by the shell, i.e. by the circuit, at P, then the scalar potential is taken as

$$\phi = i\omega, \text{ giving } \mathbf{H} = -\operatorname{grad}\phi = -i\operatorname{grad}\omega \quad . \qquad (8.9)$$

We now proceed to show that this agrees with the vector potential.

Give the point P a small vector displacement $d\mathbf{p}$ in a sense to increase $\mathbf{r}$, the vector joining P to an element $d\mathbf{l}$ of the circuit; this is equivalent to fixing P and moving the circuit bodily in the opposite sense to the dotted position. The element sweeps out a vector area $d\mathbf{l} \times d\mathbf{p}$ which subtends a solid angle $\mathbf{r}_1' \cdot d\mathbf{l} \times d\mathbf{p}/r^2$ at P, where $\mathbf{r}_1'$ is unit vector in the sense from P to the element. The integral of this taken round the whole circuit is the total change in solid angle subtended by the circuit when it is slightly displaced. Applying the rules for scalar triple products, Equation 2.21, gives

$$\frac{\mathbf{r}_1'}{r^2}\cdot(d\mathbf{l} \times d\mathbf{p}) = d\mathbf{p}\cdot\left(\frac{\mathbf{r}_1'}{r^2} \times d\mathbf{l}\right) = \mathbf{p}_1 dp\cdot\left(\frac{\mathbf{r}_1'}{r^2} \times d\mathbf{l}\right),$$

if $\mathbf{p}_1$ is unit vector along $d\mathbf{p}$. Dividing by dp and integrating round the circuit gives

$$\frac{d\omega}{dp} = \mathbf{p}_1\cdot\oint\frac{\mathbf{r}_1'}{r^2} \times d\mathbf{l}\,;$$

since P moves away from the circuit this is the rate of *decrease* of solid angle subtended by the circuit. But grad ω is the total rate of *increase* of ω ; hence $-\mathbf{p}_1\cdot$grad ω is the component rate of decrease in any direction denoted by $\mathbf{p}_1$, equivalent to $d\omega/dp$ above. Also if $\mathbf{r}_1$ is unit vector drawn towards P from $d\mathbf{l}$, $\mathbf{r}_1' = -\mathbf{r}_1$. Taking the scalar product with $\mathbf{p}_1$ in Equation 8.9,

$$\mathbf{H}\cdot\mathbf{p}_1 = -i\mathbf{p}_1\cdot\text{grad}\,\omega = i\frac{d\omega}{dp} = -\mathbf{p}_1\cdot i\oint\frac{(\mathbf{r}_1 \times \mathbf{t})}{r^2}\,dl.$$

For this to be true of any displacement,

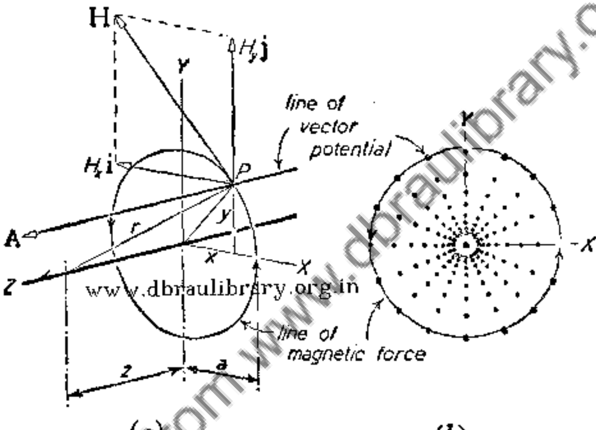
$$\mathbf{H} = -i\oint\frac{(\mathbf{r}_1 \times \mathbf{t})\,dl}{r^2},$$

which is Equation 8.8. Hence the scalar magnetic potential defined by Equation 8.9 gives the same magnetic field as the vector potential of Equation 8.7; we can write, therefore,

$$\mathbf{H} = \text{curl}\,\mathbf{A} = -\text{grad}\,\phi,$$

where $\mathbf{A} = i\oint\frac{d\mathbf{l}}{r}$ and $\phi = i\omega$, proving the equivalence of the current and the shell in the field produced at any point.

5. Simple Examples of Vector Potential. The idea of vector potential is more readily grasped by considering simple examples of linear currents. First take the case of an infinitely long, thin wire carrying current along the positive direction of Z as shown in Fig. 34(a). The lines of magnetic force are circles with their centres on the Z axis, the circles lying in planes parallel to the plane of XY; in other words, since the wire is infinitely long, the field distribution in the plane of XY is precisely similar to that



(a) (b)

FIG. 34.—Vector Potential of Straight Current

in any other parallel plane. Since all the elements of current are in the same direction, the vector potential at any point P is a vector parallel to OZ and can be found by a simple scalar integration. Writing $d\mathbf{l} = \mathbf{k}\,dz$ and $r^2 = z^2 + a^2$, Equation 8.7 gives

$$\mathbf{A} = i\mathbf{k}\int_{-\infty}^{\infty}\frac{dz}{(z^2+a^2)^{\frac{1}{2}}} = i\mathbf{k}\Big[{}_{-\infty}^{\infty}\ \text{arc sinh}\,(z/a)\Big]$$

$$= 2i\mathbf{k}\Big[{}_{0}^{\infty}\ \text{arc sinh}(z/a)\Big]$$

$$= 2i\mathbf{k}\Big\{{}_{0}^{\infty}\ \log[z+\sqrt{(z^2+a^2)}]\Big\} = (C_\infty - 2i\log a)\mathbf{k},$$

where C_∞ is an infinitely great constant arising from the infinite extent of the 'circuit' round which the integral has been taken. Since we are concerned with derivatives

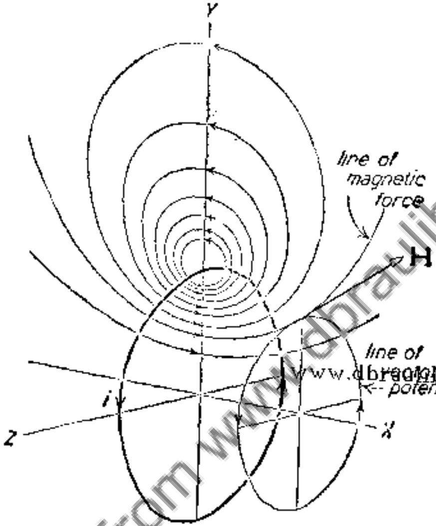


FIG. 35.—Vector Potential of Circular Current

of **A**, the presence of this constant is immaterial; without loss of generality we can take

$$\mathbf{A} = A_z\mathbf{k} = -2i\mathbf{k}\log a = -i\mathbf{k}\log(x^2+y^2).$$

Since A_x and A_y are zero, the usual expressions for the components of curl **A** give, from Equation 4.9,

$$H_x = \partial A_z/\partial y = -2iy/(x^2+y^2) = -(2i/a)\sin\theta,$$
$$H_y = -\partial A_z/\partial x = 2ix/(x^2+y^2) = (2i/a)\cos\theta,$$
$$H_z = 0. \quad \text{(cf. p 75.)}$$

Surfaces of constant vector potential are coaxial cylinders; with equal increments in passing from one to the next their radii increase in geometric progression. Lines of vector potential are, therefore, closely packed together near the wire and fall in density at greater distances as shown by Fig. 34(*b*).

The corresponding multivalued scalar potential is easily found. As in Fig. 30 (*b*) take the right-hand half plane of ZX as the equivalent shell (p. 77). Draw a sphere of unit radius with centre at P; then the half plane subtends an area $4\pi(\pi - \theta)/2\pi = 2(\pi - \theta)$ at the spherical surface. This is the solid angle subtended by the plane and Equation 8.9 gives $\phi = 2i(\pi - \theta)$ as on p. 77. The components of $-$ grad ϕ, i.e. of $\mathbf{H}$, are identical with those of curl $\mathbf{A}$ on p. 89, as the reader should verify by using Equation 4.3 and $\theta = \text{arc tan } (y/x)$.

Another simple case is the current in a circle of wire, as shown in Fig. 35, the circle lying in the plane of YZ. At any point on the axis of X the vector potential is zero, the point being equidistant from all elements of the circle. For a point off the axis, nearer therefore to one side of the circle, it is geometrically obvious that the resultant vector potential due to the whole circle is a vector parallel to the tangent at the nearest part. Let the point describe a circle about the axis, parallel to the current; then by symmetry this argument applies to any point on this locus. Hence the lines of vector potential for a circular current filament are parallel circles with their centres on the common axis. To give an expression for $\mathbf{A}$ and hence for $\mathbf{H}$ requires the use of elliptic functions and is beyond our present discussion.

The diagram, Fig. 35, represents also in one meridional plane the lines of flow of fluid about a circular vortex filament; the complete picture is obtained by revolving the diagram about OX.

CHAPTER IX

THE ELECTROMAGNETIC FIELD EQUATIONS OF MAXWELL

1. General. The most striking example of the success of vector methods is provided by the theory of the electromagnetic field. Although Maxwell introduced the fundamental vector ideas into the theory, he did not make any use of vector analysis in manipulating his equations, relying entirely upon the usual lengthy cartesian methods. It is to Heaviside, Lorentz and many later workers that we must turn for the exclusive adoption of the concise vector equations and their treatment by purely vectorial methods.

Our present object is to state the necessary equations and to use them as an exercise in the application of principles established in earlier chapters. We shall not, therefore, be concerned with questions of a purely physical nature, for the discussion of which the reader is referred to the special works mentioned in the Bibliography.* One main assumption will be made throughout this chapter, namely, that the medium sustaining the field is homogeneous, i.e. composed of one kind of material in the region considered, and also isotropic, i.e. having identical physical properties in all directions at every point.

2. Maxwell's Equations. At any instant let **c** be the current density and **H** the magnetic force at any point in the field, both being expressed in electromagnetic units.

* See particularly F. W. G. White, *Electromagnetic Waves* (Methuen, 1934,) in this series of Monographs. Also see p. 118.

Then by the method of p. 80 the circuital or magnetomotive force theorem, Equation 8.1 gives

$$\text{curl } \mathbf{H} = 4\pi\mathbf{c}.$$

In deducing this relation, however, a purely conducting medium was assumed, **c** being the actual flow of electric charges per second normally through unit area. Maxwell observed, however, that electromagnetic phenomena take place in dielectric non-conductors, even in vacuous space, in which electric flow is impossible. To account for this he postulated a vector **D**, known as the *electric displacement* at any point and having the physical dimensions of charge per unit area, the time rate of change of which is called the *displacement current density* at the point. He regards the electric force $\mathcal{E}$, which is the force per unit charge tending to move electricity at any point, as related to the displacement **D** in much the same way as stress is related to strain in an elastic solid. Using electromagnetic units, if κ is the dielectric constant of the medium,

$$\mathbf{D} = \kappa\,\mathcal{E}/4\pi \quad . \quad . \quad . \quad . \quad . \quad (9.1)$$

In general a semi-conductor will exhibit both phenomena, i.e. true flow of electricity, the conduction current, together with displacement or, as an engineer would say, charging current. Writing the total current density as

$$\mathbf{C} = \mathbf{c} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{c} + \frac{\kappa}{4\pi}\frac{\partial\,\mathcal{E}}{\partial t}, \quad . \quad . \quad . \quad (9.2)$$

the first equation must be amended to read

$$\text{curl } \mathbf{H} = 4\pi\mathbf{C} = 4\pi(\mathbf{c} + \dot{\mathbf{D}}) = 4\pi\mathbf{c} + \kappa\dot{\mathcal{E}},$$

using the dot notation as an abbreviation for time differentiation. Maxwell's *assumption* that a displacement current gives rise to a magnetic force in the same way as a conduction current was a stroke of genius that led him to the theory of electromagnetic waves in space and the electromagnetic nature of light; his conclusions were not verified by experiment until many years later. The total current flows in a closed circuit, i.e. it is solenoidal and

$$\text{div } \mathbf{C} = 0 \quad . \quad . \quad . \quad . \quad . \quad (9.3)$$

Again draw in the field any closed curve and calculate the line integral of the electric force round it. This is the electromotive force round the curve and by Stokes's theorem can be transformed into the surface integral of curl $\mathcal{E}$ taken over any open surface with the curve as its edge. But by Faraday's law the electromotive force is the rate at which the magnetic flux through the surface is decreasing. Let $\mathbf{B} = \mu\mathbf{H}$ be the magnetic induction or flux density, μ being the permeability, then

$$\oint \mathcal{E}\cdot d\mathbf{l} = \iint \mathbf{n}\cdot\text{curl } \mathcal{E}\, ds = -\frac{\partial}{\partial t}\iint \mathbf{n}\cdot\mathbf{B}\, ds,$$

which to be true for any surface bounded by the curve requires

$$\text{curl } \mathcal{E} = -\dot{\mathbf{B}} = -\mu\dot{\mathbf{H}}.$$

The lines of magnetic induction are necessarily solenoidal, i.e.

$$\text{div } \mathbf{B} = \mu \text{ div } \mathbf{H} = 0.$$

Summarizing, Maxwell's equations for the field are

$$\text{curl } \mathbf{H} = 4\pi\mathbf{C} = 4\pi\mathbf{c} + \kappa\dot{\mathcal{E}}, \quad . \quad . \quad . \quad (9.4)$$

$$\text{curl } \mathcal{E} = -\dot{\mathbf{B}} = -\mu\dot{\mathbf{H}}; \quad . \quad . \quad . \quad (9.5)$$

together with

$$\mathbf{C} = \mathbf{c} + \dot{\mathbf{D}}, \quad . \quad . \quad . \quad . \quad . \quad (9.2\ bis)$$

$$\text{div } \mathbf{B} = 0,$$

and

$$\text{div } \mathbf{D} = \varrho,$$

ϱ being the volume density of electric charge at any point. These general equations will be returned to in Section 5.

3. Conducting Media. When the medium is a conductor there is no displacement current and the main equations become

$$\text{curl } \mathbf{H} = 4\pi\mathbf{c} \text{ and } \text{curl } \mathcal{E} = -\dot{\mathbf{B}}.$$

If γ is the electrical conductivity of the medium, then $\mathbf{c} = \gamma\boldsymbol{\mathcal{E}}$, by Ohm's law; the equations can now be written

$$\text{curl } \mathbf{B} = 4\pi\mu\mathbf{c} \text{ and curl } \mathbf{c} = -\gamma\dot{\mathbf{B}}.$$

Taking the curl of the first, Equation 5.7 gives

$$\text{curl curl } \mathbf{B} = \text{grad div } \mathbf{B} - \nabla^2\mathbf{B} = 4\pi\mu \text{ curl } \mathbf{c} = -4\pi\mu\gamma\dot{\mathbf{B}}.$$

But div **B** is zero, so that the equation for **B** is

$$\nabla^2\mathbf{B} = 4\pi\mu\gamma\dot{\mathbf{B}} \quad . \quad . \quad . \quad . \quad (9.6)$$

Similarly, by curling the second equation and noting that div **c** vanishes,

$$\nabla^2\mathbf{c} = 4\pi\mu\gamma\dot{\mathbf{c}} \quad . \quad . \quad . \quad . \quad . \quad (9.7)$$

These equations have been solved for many practical cases, e.g. eddy current induction in wires, bars or plates.

They can be conveniently put in terms of vector potential by defining

$$\text{curl } \mathbf{A} = \mathbf{B}, \quad . \quad . \quad . \quad . \quad . \quad (9.8)$$

together with curl $\mathbf{B} = 4\pi\mu\mathbf{c}$ and curl $\mathbf{c} = -\gamma\dot{\mathbf{B}}$.
Now take the curl of curl **A** and choose **A** to make div **A** zero; then

$$\nabla^2\mathbf{A} = -4\pi\mu\mathbf{c} \quad . \quad . \quad . \quad . \quad . \quad (9.9)$$

The third equation makes

$$\mathbf{c} = -\gamma\dot{\mathbf{A}} \quad . \quad . \quad . \quad . \quad . \quad (9.10)$$

Eliminating **c**

$$\nabla^2\mathbf{A} = 4\pi\mu\gamma\dot{\mathbf{A}} \quad . \quad . \quad . \quad . \quad . \quad (9.11)$$

4. Dielectric Media. In a dielectric there can be no conduction current, so that Equations 9.4 and 9.5 become simply

$$\text{curl } \mathbf{H} = \kappa\dot{\boldsymbol{\mathcal{E}}} \text{ and curl } \boldsymbol{\mathcal{E}} = -\mu\dot{\mathbf{H}}.$$

Curling the first,

$$\text{curl curl } \mathbf{H} = \text{grad div } \mathbf{H} - \nabla^2\mathbf{H} = \kappa \text{ curl } \dot{\boldsymbol{\mathcal{E}}} = -\kappa\mu(\partial^2\mathbf{H}/\partial t^2).$$

Since div $\mathbf{H} = 0$, this can be written

$$\left(\nabla^2 - \kappa\mu\frac{\partial^2}{\partial t^2}\right)\mathbf{H} = \text{dal } \mathbf{H} = 0, \quad . \quad . \quad (9.12)$$

with precisely similar forms for **B**, **ℰ** and **D**. The bracketed operator is known as the d'Alembertian, after the French mathematician d'Alembert, who first used a particular form of it in the theory of wave motion along elastic strings. It is not difficult to show that Equation 9.12 indicates that the components of **H** (and those of the other three quantities also) are propagated in the medium with a finite velocity $1/\sqrt{(\kappa\mu)}$. In free space this is the velocity of light, a fact which led Maxwell to the conclusion that light is itself an electromagnetic phenomenon.

5. Energy Considerations. Calculation of the energy conditions in an electromagnetic field provides an excellent example of vector methods. Returning to the general case of a semi-conducting dielectric, consider an element of volume dv in which the electric and magnetic forces are **ℰ** and **H** respectively and the conduction current density is **c**. Then the Joule heat generated per second within the element is

$$dP = \boldsymbol{\mathcal{E}}\cdot\mathbf{c}\,dv = (\mathcal{E}_x c_x + \mathcal{E}_y c_y + \mathcal{E}_z c_z)\,dx\,dy\,dz,$$

the cartesian form enabling the reader to verify the vectorial statement. Now Equation 9.4 makes

$$\mathbf{c} = \frac{1}{4\pi}\,\text{curl}\,\mathbf{H} - \dot{\mathbf{D}},$$

and
$$dP = \left[\frac{1}{4\pi}\boldsymbol{\mathcal{E}}\cdot\text{curl}\,\mathbf{H} - \boldsymbol{\mathcal{E}}\cdot\dot{\mathbf{D}}\right]dv.$$

Equation 4.12 gives

$$\text{div}\,(\boldsymbol{\mathcal{E}}\times\mathbf{H}) = \mathbf{H}\cdot\text{curl}\,\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}\cdot\text{curl}\,\mathbf{H},$$

so that
$$dP = \left[\frac{1}{4\pi}\mathbf{H}\cdot\text{curl}\,\boldsymbol{\mathcal{E}} - \frac{1}{4\pi}\,\text{div}\,(\boldsymbol{\mathcal{E}}\times\mathbf{H}) - \boldsymbol{\mathcal{E}}\cdot\dot{\mathbf{D}}\right]dv.$$

Using the second main equation, 9.5, and also Equation 9.1,

$$dP = -\frac{1}{4\pi}[\mu\mathbf{H}\cdot\dot{\mathbf{H}} + \kappa\,\boldsymbol{\mathcal{E}}\cdot\dot{\boldsymbol{\mathcal{E}}} + \text{div}\,(\boldsymbol{\mathcal{E}}\times\mathbf{H})]\,dv.$$

But it is easy to see that Equation 3.3a makes

$$\mathbf{H}\cdot\dot{\mathbf{H}} = \tfrac{1}{2}\frac{\partial}{\partial t}(\mathbf{H}\cdot\mathbf{H}) = \tfrac{1}{2}\frac{\partial}{\partial t}(H^2) \text{ and } \boldsymbol{\mathcal{E}}\cdot\dot{\boldsymbol{\mathcal{E}}} = \tfrac{1}{2}\frac{\partial}{\partial t}(\mathcal{E}^2).$$

Also Gauss's divergence theorem, Equation 6.1, transforms the volume integral

$$\iiint \operatorname{div}(\boldsymbol{\mathcal{E}} \times \mathbf{H})\, dv \text{ into} \iint \mathbf{n}\cdot(\boldsymbol{\mathcal{E}} \times \mathbf{H})\, ds,$$

the surface integral of the normal component of $\boldsymbol{\mathcal{E}} \times \mathbf{H}$ taken over the surface enclosing a given volume of the field. Finally

$$\iiint\left\{dP + \frac{\partial}{\partial t}\left[\frac{\mu H^2}{8\pi} + \frac{\kappa\mathcal{E}^2}{8\pi}\right]\right\}dv = -\frac{1}{4\pi}\iint \mathbf{n}\cdot(\boldsymbol{\mathcal{E}} \times \mathbf{H})\, ds, \quad (9.13)$$

which is *Poynting's theorem.*

On the left side we have the sum of the Joule heat generated and the magnetic and electric energies stored per second in the volume. On the right is the total inward normal flux of a vector

$$\mathbf{R} = (\boldsymbol{\mathcal{E}} \times \mathbf{H})/4\pi,$$

which is known as the *Poynting radiant vector*. It must represent the rate of flow of energy per unit area normal to $\mathbf{R}$. Note that $\mathbf{R}$ is normal to the plane containing $\boldsymbol{\mathcal{E}}$ and $\mathbf{H}$ and forms a right-handed system with these vectors. Equation 9.13 states that the sum of the energy per second wasted in heat and that stored electromagnetically in any element of volume of an e.m. field has entered that element as a flux of energy per second normally through its bounding surface.

CHAPTER X

ELEMENTARY PROPERTIES OF THE LINEAR VECTOR FUNCTION

1. The Linear Vector Function. The simplest example of a linear vector function is provided by a vector **U** which is a constant scalar multiple of a second vector **V**, represented by

$$\mathbf{U} = k\mathbf{V}, \quad . \quad . \quad . \quad . \quad . \quad (10.1)$$

k being a constant. This simply means that **U** is k times the magnitude of **V** and in the same direction. If **V** is a point function of space, so also is **U** but altered in scale k times. Familiar examples are the relations between **B** and **H** or between **D** and **ℰ** in homogeneous, isotropic media.

Rather more generally, let each component of **V** be multiplied linearly by a constant, different for each component. Using rectangular co-ordinates, if

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k} \quad . \quad . \quad . \quad (10.2)$$

and

$$\mathbf{U} = U_x\mathbf{i} + U_y\mathbf{j} + U_z\mathbf{k}, \quad . \quad . \quad . \quad (10.3)$$

then this more general type of linear vector function requires

$$U_x = k_xV_x, \ \ U_y = k_yV_y \text{ and } \ U_z = k_zV_z \quad . \quad (10.4)$$

The vector **U** is not only different in magnitude but also different in direction from the vector **V**. As examples may be mentioned the relation between **D** and **ℰ** in crystals, or that between stress and strain in a crystalline solid; indeed any case of anisotropic media in which the rectangular axes are *principal axes* of strain.

Most general of all is the case of complete anisotropy, where each component of **U** is a linear function of all three components of **V**. In rectangular co-ordinates the relation between the components can be written

$$\left.\begin{aligned} U_x &= k_{xx}V_x + k_{xy}V_y + k_{xz}V_z, \\ U_y &= k_{yx}V_x + k_{yy}V_y + k_{yz}V_z, \\ U_z &= k_{zx}V_x + k_{zy}V_y + k_{zz}V_z. \end{aligned}\right\} \quad . \quad . \quad . \quad (10.5)$$

The nine coefficients characterize the transformation of the components of one vector into those of the other; in general, $k_{\alpha\beta}$ transforms V_β into one of the three parts of U_α, giving a rule whereby to remember the significance of the subscripts. The final result of the transformation is

$$\mathbf{U} = \Phi\mathbf{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.6)$$

which may be regarded as a generalization of Equation 10.1; Φ is an operator turning **V** into **U** by a linear modification of components specified by Equation 10.5. It is rather graphically described as a *Cartesian tensor*, or simply as a *tensor*, since the relation of strain to stress in an anisotropic elastic solid obeying Hooke's law is of this kind.

The operator Φ is also sometimes called an *affinor*, for the following reason. Let **V** have the particular value of **r**, the radius vector from the origin to a point in space. Then **U** is a vector to a second point related to the first by a general linear strain; the points are said to be in 1 : 1 correspondence or in affinity.

2. Simple Types of Tensors. The essential part of a tensor operation is the array of coefficients, such as $k_{\alpha\beta}$, which in the general case are nine in number; this can be symbolically represented, writing the array in the form of a *matrix* *, thus

$$\Phi = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad (10.7)$$

* More correctly a *square matrix*. The general type of matrix has *mn* elements in *m* rows of *n* columns. It represents the array of coefficients in *m* linear equations with *n* variables and is of a much more general type than we have space to examine here.

Tensors and their matrices are manipulated by certain rules, which are chosen to give results consistent with the algebra of groups of linear equations, such as Equation 10.5. A matrix is a convenient symbol for a whole group and enables the set of equations to be handled as a single entity. It is beyond our purpose to deal in any detail with matrix algebra, but one or two simple rules will be stated. First, if $\mathbf{V} = \mathbf{A} + \mathbf{B}$ we define

$$\Phi(\mathbf{A} + \mathbf{B}) = \Phi\mathbf{A} + \Phi\mathbf{B};$$

i.e. the tensor operation is regarded as following the same law of distribution as for multiplication in scalar algebra. Second, if Φ_1 and Φ_2 are two different tensors,

$$\Phi_1\mathbf{V} + \Phi_2\mathbf{V} = \Phi_2\mathbf{V} + \Phi_1\mathbf{V} = (\Phi_1 + \Phi_2)\mathbf{V};$$

i.e. tensors follow the commutative law for addition. This the reader may readily verify by setting down the sum of two sets of equations like Equation 10.5, when it will be seen that the sum of two matrices is a new matrix in which the elements are the sum of the elements of Φ_1 and Φ_2. Third, a tensor is regarded as negative when every coefficient in its matrix is reversed in sign.

If the columns and rows of a matrix are interchanged, the resulting tensor is the *conjugate tensor*, for which the matrix is

$$\Phi_c = \begin{bmatrix} k_{xx} & k_{yx} & k_{zx} \\ k_{xy} & k_{yy} & k_{zy} \\ k_{xz} & k_{yz} & k_{zz} \end{bmatrix} \quad . \quad . \quad . \quad . \quad (10.8)$$

When two tensors are such that one is the conjugate of the other, inspection of Equations 10.7 and 10.8 shows that

$$k_{xy} = k_{yx},\ k_{xz} = k_{zx} \text{ and } k_{yz} = k_{zy} \quad . \quad . \quad (10.9)$$

If a single tensor satisfies this condition it is called *symmetrical* or *self-conjugate*. It has only *six* independent elements and may be written

Our work is limited to the cartesian tensor and its square matrix, for which $m = n = 3$.

$$\Phi_{sym} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz} \end{bmatrix} \quad . \quad . \quad . \quad (10.10)$$

Symmetrical tensors are of frequent occurrence in physical problems and their meaning will be adverted to in Section 3.

In particular when the relation between **U** and **V** is referred to principal axes all coefficients are zero except the diagonal k_{xx}, k_{yy}, k_{zz}, corresponding to Equation 10.4.

When two tensors are such that one is minus the conjugate of the other, the coefficients must be such that

$$k_{xx}=k_{yy}=k_{zz}=0, \; k_{xy}=-k_{yx}, \; k_{xz}=-k_{zx} \text{ and } k_{yz}=-k_{zy} \quad (10.11)$$

If a single tensor satisfies this condition it is called *anti-symmetrical*, *anti-self-conjugate* or *skew*. It has only *three* coefficients and its matrix is

$$\Phi_{sk} = \begin{bmatrix} 0 & k_{xy} & k_{xz} \\ -k_{xy} & 0 & k_{yz} \\ -k_{xz} & -k_{yz} & 0 \end{bmatrix} \quad . \quad . \quad . \quad (10.12)$$

Since a skew tensor shares with a vector the property of having only three components, the operation of Φ_{sk} on a vector **V** is exactly equivalent to the vector product of two vectors, since the final result is itself a vector **U**.

3. The Symmetrical Tensor. The symmetrical tensor, Equation 10.10, has a simple geometric interpretation that will now be explained. Let **i**, **j**, **k** in Fig. 36 be unit vectors along rectangular axes X, Y, Z. With the same origin let **a**, **b**, **c** be unit vectors along a second set of rectangular axes A, B, C. The two systems are related by the cosines of the nine angles between all possible pairs of unit vectors, one in each system; these nine cosines may be represented by the matrix,

$$\begin{array}{c} \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{array} \\ \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \end{array}$$

e.g. l_1 is the cosine of the angle between **i** and **a**, &c. Since both systems are rectangular, it is clear that only three of the cosines can be independent, i.e. one system can be put into the position of the other by three independent angular movements. Project the **i**, **j**, **k** system into the directions of **a**, **b**, **c**; then it is geometrically obvious that

$$\mathbf{a} = l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k}$$
$$\mathbf{b} = m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k}$$
$$\mathbf{c} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}.$$

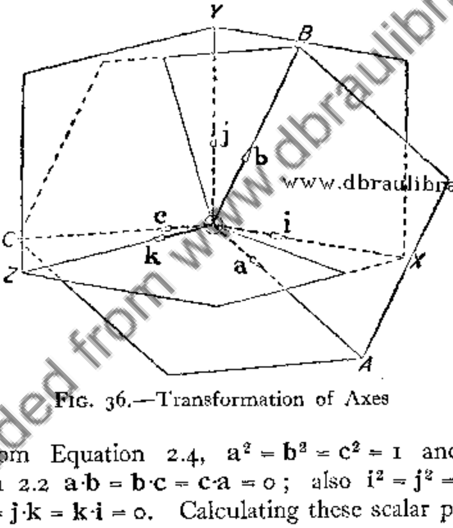


FIG. 36.—Transformation of Axes

Now from Equation 2.4, $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$ and from Equation 2.2 $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$; also $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$. Calculating these scalar products gives

$$l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1,$$
$$l_1m_1 + l_2m_2 + l_3m_3 = m_1n_1 + m_2n_2 + m_3n_3 = n_1l_1 + n_2l_2 + n_3l_3 = 0,$$

which are the well-known six relations between the cosines; showing that only three can be independently chosen.

If the relation of **U** to **V** is given in one set of axes, what is it with reference to the other? Let the **a**, **b**, **c** system be selected in such a way that they are principal axes; then, as defined by Equation 10.4,

$$U_a = k_a V_a, \ U_b = k_b V_b \text{ and } U_c = k_c V_c$$

is the relation between the components along A, B and C respectively. Now resolve V_x, V_y, V_z, the components of **V** in the X, Y, Z system, into the other system, thus

$$\left.\begin{aligned} V_a &= l_1 V_x + l_2 V_y + l_3 V_z \\ V_b &= m_1 V_x + m_2 V_y + m_3 V_z \\ V_c &= n_1 V_x + n_2 V_y + n_3 V_z \end{aligned}\right\} \text{ and } \begin{aligned} U_a &= k_a V_a \\ U_b &= k_b V_b \\ U_c &= k_c V_c \end{aligned}$$

Now resolve U_a, U_b, U_c upon X, giving

$$\begin{aligned} U_x = l_1 U_a + m_1 U_b + n_1 U_c = (k_a l_1^2 + k_b m_1^2 + k_c n_1^2) V_x \\ + (k_a l_1 l_2 + k_b m_1 m_2 + k_c n_1 n_2) V_y \\ + (k_a l_1 l_3 + k_b m_1 m_3 + k_c n_1 n_3) V_z \end{aligned}$$

Likewise resolving upon Y and Z,

$$\begin{aligned} U_y = (k_a l_1 l_2 + k_b m_1 m_2 + k_c n_1 n_2) V_x \\ + (k_a l_2^2 + k_b m_2^2 + k_c n_2^2) V_y \\ + (k_a l_2 l_3 + k_b m_2 m_3 + k_c n_2 n_3) V_z, \end{aligned}$$

$$\begin{aligned} U_z = (k_a l_1 l_3 + k_b m_1 m_3 + k_c n_1 n_3) V_x \\ + (k_a l_2 l_3 + k_b m_2 m_3 + k_c n_2 n_3) V_y + (k_a l_3^2 + k_b m_3^2 + k_c n_3^2) V_z. \end{aligned}$$

Comparing these three relations with Equations 10.5 it will be found that the coefficients satisfy Equation 10.9; the tensor relating **U** to **V** in the system of axes X, Y, Z is, therefore, symmetrical like Equation 10.10. It follows that any symmetrical tensor corresponds with a transformation from the principal axes to another rectangular system of reference.

It is possible to give a simple graphical construction for a symmetrical linear function. Using Equation 10.10 the function is

$$\mathbf{U} = \Phi_{sym} \mathbf{V}.$$

Now project **V** upon **U**, i.e. calculate by Equation 2.10

$$\mathbf{V} \cdot \mathbf{U} = V_x U_x + V_y U_y + V_z U_z.$$

Using the symmetry of Equation 10.10 in Equation 10.5,

$$\mathbf{V}\cdot\mathbf{U} = \mathbf{V}\cdot\Phi_{sym}\mathbf{V} = k_{xx}V_x^2 + k_{yy}V_y^2 + k_{zz}V_z^2 + 2[k_{xy}V_xV_y + k_{yz}V_yV_z + k_{xz}V_zV_x] \equiv S,$$

say, where S is a scalar point-function. But

$$U_x = k_{xx}V_x + k_{xy}V_y + k_{xz}V_z = \tfrac{1}{2}\frac{\partial S}{\partial V_x}$$

and similarly $U_y = \frac{1}{2}\frac{\partial S}{\partial V_y}$ and $U_z = \frac{1}{2}\frac{\partial S}{\partial V_z}$.

But these are the components of $\frac{1}{2}$ grad S in terms of V_x, V_y and V_z as co-ordinates, Equation 4.3, so that

$$\mathbf{U} = U_x\mathbf{i} + U_y\mathbf{j} + U_z\mathbf{k} = \tfrac{1}{2}\text{ grad } S; \quad . \quad (10.13)$$

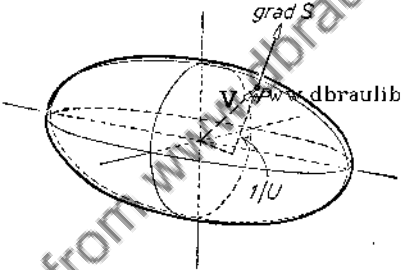


FIG. 37.—The Tensor Ellipsoid

it follows, therefore, that $\mathbf{U}$ is a vector perpendicular to the surface $S = constant$ in the direction of the outward normal. But $S = constant$ is an equation of the second degree in the rectangular components of $\mathbf{V}$; regarding these as co-ordinates defining the extremity P of the vector $\mathbf{V}$, the locus of that extremity is a quadric surface or conicoid. In particular, if $S = 1$ the surface is, under certain conditions, the *tensor ellipsoid*, and

$$\mathbf{V}\cdot\mathbf{U} = (\text{resolute of } \mathbf{V} \text{ in direction of } \mathbf{U})U = 1 \quad (10.14)$$

Hence this resolute in the direction of grad S is the reciprocal of the magnitude U of the vector $\mathbf{U}$. Fig. 37 shows this geometric relation for a tensor ellipsoid set in the direction of its principal axes.

4. Resolution of a Tensor. Return now to the general tensor, Equation 10.7, and apply the distributive law to it and its conjugate; then

$$\Phi = \tfrac{1}{2}(\Phi + \Phi_c) + \tfrac{1}{2}(\Phi - \Phi_c) \quad . \quad . \quad (10.15)$$

If the reader will write down the matrices for $\Phi + \Phi_c$ and $\Phi - \Phi_c$ by adding corresponding coefficients, it is easy to see that the former is a symmetrical and the latter a skew tensor. Hence any tensor relation between two vectors is equivalent to the sum of a symmetrical relation and a skew relation. The first corresponds to a change of axes without deformation of the system of reference, as has been shown, and the second to a transformation of the angles between the axes themselves.

5. Repeated Tensor Operations. In vector theory we frequently need to express relations between more than two linearly connected vectors in an anisotropic medium. Thus in electromagnetism there is a tensor relation between **H** and **B**, and a second relation between **B** and the ponderomotive force on magnetized solids. In a general way, if

$$\mathbf{U} = \Phi_1\mathbf{V} \text{ where } \Phi_1 = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix},$$

$$\text{and} \quad \mathbf{V} = \Phi_2\mathbf{W} \text{ where } \Phi_2 = \begin{bmatrix} c_{xx} & c_{xy} & c_{xz} \\ c_{yx} & c_{yy} & c_{yz} \\ c_{zx} & c_{zy} & c_{zz} \end{bmatrix},$$

what is the meaning of

$$\mathbf{U} = \Phi_1\Phi_2\mathbf{W}\,?$$

The answer is easily found by writing out the three relations between the components of **U** and those of **V**, Equation 10.5, and the analogous set giving the connexion between the components of **V** and those of **W**. From the second set of equations substitute the components of **V** in the first set, giving **U** in terms of **W**. The final matrix for the complete transformation can be verified by the reader to be $\Phi_1\Phi_2 =$

$$\begin{bmatrix} (k_{xx}c_{xx}+k_{xy}c_{yx}+k_{xz}c_{zx}) & (k_{xx}c_{xy}+k_{xy}c_{yy}+k_{xz}c_{zy}) & (k_{xx}c_{xz}+k_{xy}c_{yz}+k_{xz}c_{zz}) \\ (k_{yx}c_{xx}+k_{yy}c_{yx}+k_{yz}c_{zx}) & (k_{yx}c_{xy}+k_{yy}c_{yy}+k_{yz}c_{zy}) & (k_{yx}c_{xz}+k_{yy}c_{yz}+k_{yz}c_{zz}) \\ (k_{zx}c_{xx}+k_{zy}c_{yx}+k_{zz}c_{zx}) & (k_{zx}c_{xy}+k_{zy}c_{yy}+k_{zz}c_{zy}) & (k_{zx}c_{xz}+k_{zy}c_{yz}+k_{zz}c_{zz}) \end{bmatrix} \tag{10.16}$$

An inspection of this apparently complex matrix will show that its construction follows a very simple rule: The element in the *r*th horizontal row and *c*th vertical column of $\Phi_1\Phi_2$ is obtained by multiplying each element in the *r*th row of Φ_1 by the corresponding element in the *c*th column of Φ_2 and adding the results. It is usual formally to define $\Phi_1\Phi_2$ as the 'product' of the two matrices, the rule for the 'multiplication' being that just stated. It is obvious that $\Phi_2\Phi_1$ will, in general, be quite different from $\Phi_1\Phi_2$; hence the rule for matrix 'multiplication', as also for successive tensor operations, is not commutative. The algebra of tensors and their matrices, therefore, fails in the commutative law for multiplication exactly as we have earlier seen to be true of vectors.

A matrix of particular interest is

$$v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which, as the reader can verify, has the important property that

$$v\Phi = \Phi v = \Phi.$$

Hence v has the same effect in tensor algebra as unity in ordinary scalar algebra, since it leaves a vector unchanged, and is called the *unit tensor* or *unit matrix*.

6. The Dyadic. Although tensors are widely used to express linear vector relationships in an anisotropic field, there is another method by which this can be done. The dyadic, introduced by Gibbs as an essential part of his treatment of vector analysis, is preferred by some writers, and its use will now be briefly explained.

Returning to Equation 10.3 and inserting the values of U_x, U_y and U_z given by Equation 10.5, the expression for $\mathbf{U}$ can be written as

$$\mathbf{U} = (k_{xx}\mathbf{i}+k_{yx}\mathbf{j}+k_{zx}\mathbf{k})V_x+(k_{xy}\mathbf{i}+k_{yy}\mathbf{j}+k_{zy}\mathbf{k})V_y+(k_{xz}\mathbf{i}+k_{yz}\mathbf{j}+k_{zz}\mathbf{k})V_z$$

$$= \mathbf{l}V_x+\mathbf{m}V_y+\mathbf{n}V_z$$

say. Using Equation 2.9, $V_x = \mathbf{i}\cdot\mathbf{V}$, $V_y = \mathbf{j}\cdot\mathbf{V}$ and $V_z = \mathbf{k}\cdot\mathbf{V}$, giving

$$\mathbf{U}=\mathbf{li}\cdot\mathbf{V}+\mathbf{mj}\cdot\mathbf{V}+\mathbf{nk}\cdot\mathbf{V}=(\mathbf{li}+\mathbf{mj}+\mathbf{nk})\cdot\mathbf{V}=\Psi\cdot\mathbf{V} \quad . \quad . \quad (10.17)$$

The bracketed expression is known as a *dyadic* and consists of the sum of three terms, **li**, **mj**, **nk**, which are its *dyads*. Each dyad is made up of two vectors in succession, the first vector being called the *antecedent* of the dyad (such as **l**, **m**, **n**) and the second vector its *consequent* (such as **i**, **j**, **k**). It is to be noted that a dyad is not a simple product, but merely a symbolic juxtaposition of two vectors. In interpreting the operator $\Psi\cdot$ the vector nearest to **V** in any dyad is involved in a scalar product with **V**, this product then being a scalar multiplier for the remaining vector of the dyad. Comparing Equation 10.17 with 10.6, we see that the operation of a dyadic on a vector by means of this scalar product rule is equivalent to the operation with a cartesian tensor.

In Equation 10.17, if we interchange Ψ and **V**,

$$\mathbf{U}'=\mathbf{V}\cdot\mathbf{li}+\mathbf{V}\cdot\mathbf{mj}+\mathbf{V}\cdot\mathbf{nk}=\mathbf{V}\cdot(\mathbf{li}+\mathbf{mj}+\mathbf{nk})=\mathbf{V}\cdot\Psi \quad . \quad . \quad (10.18)$$

which is a linear vector function different from **U** and called its *conjugate*. In Equation 10.17 the dyadic appears as a *prefactor* and in Equation 10.18 as a *postfactor*.

Starting again from

$$\mathbf{U} = V_x\mathbf{l} + V_y\mathbf{m} + V_z\mathbf{n}$$

and commutating the scalar products for V_x, V_y, V_z gives the identity

$$\mathbf{U} = \mathbf{V}\cdot\mathbf{il} + \mathbf{V}\cdot\mathbf{jm} + \mathbf{V}\cdot\mathbf{kn} = \mathbf{V}\cdot(\mathbf{il} + \mathbf{jm} + \mathbf{kn}) = \mathbf{V}\cdot\Psi_c = \Psi\cdot\mathbf{V} \quad (10.19)$$

The dyadic Ψ_c, in which the rôles of antecedent and consequent are interchanged, is the *conjugate dyadic* of Ψ. Hence any dyadic used as a prefactor is equivalent to its conjugate used as a postfactor. If a dyadic is such that $\Psi\cdot\mathbf{V} = \mathbf{V}\cdot\Psi$ it is *self-conjugate*; if $\Psi\cdot\mathbf{V} = -\mathbf{V}\cdot\Psi$ it is *anti-self-conjugate*. These definitions are exactly parallel with those for the corresponding tensors on p. 100.

A dyadic can easily be expressed in terms of the nine possible dyads of the unit vectors **i**, **j**, **k**, and in this so-called *nonion* form the relation between dyadic, tensor and matrix are even more clearly displayed. On p. 106 substitute in **U** for V_x, V_y, V_z in terms of the unit vectors, then

$$\begin{aligned}\Psi = &(k_{xx}\mathbf{ii} + k_{yx}\mathbf{ji} + k_{zx}\mathbf{ki}) \\ &+ (k_{xy}\mathbf{ij} + k_{yy}\mathbf{jj} + k_{zy}\mathbf{kj}) \\ &+ (k_{xz}\mathbf{ik} + k_{yz}\mathbf{jk} + k_{zz}\mathbf{kk}) \quad . \quad . \quad (10.20)\end{aligned}$$

and also

$$\begin{aligned}\Psi_c = [&(k_{xx}\mathbf{ii} + k_{yx}\mathbf{ij} + k_{zx}\mathbf{ik}) \quad = \quad [(k_{xx}\mathbf{ii} + k_{xy}\mathbf{ji} + k_{xz}\mathbf{ki}) \\ &+ (k_{xy}\mathbf{ji} + k_{yy}\mathbf{jj} + k_{zy}\mathbf{jk}) \quad + (k_{yx}\mathbf{ij} + k_{yy}\mathbf{jj} + k_{yz}\mathbf{kj}) \\ &+ (k_{xz}\mathbf{ki} + k_{yz}\mathbf{kj} + k_{zz}\mathbf{kk})] \quad + (k_{zx}\mathbf{ik} + k_{zy}\mathbf{jk} + k_{zz}\mathbf{kk})] \quad (10.21)\end{aligned}$$

which are exactly analogous to the matrices, Equations 10.7 and 10.8.

If a dyadic is such that when applied to a vector, either as a prefactor or as a postfactor, it gives the vector itself, it is called the *unit dyadic* or *idemfactor*. For this to be the case all coefficients $k_{\alpha\beta}$ for which $\alpha \neq \beta$ must vanish and $k_{xx} = k_{yy} = k_{zz} = 1$. Then, as is obvious,

$$I = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} \text{ makes } I\cdot\mathbf{V} = \mathbf{V}\cdot I \equiv \mathbf{V}, \quad . \quad . \quad (10.22)$$

which is equivalent to the unit matrix, since this also corresponds with $\mathbf{U} = \mathbf{V}$. The idemfactor and unit matrix occupy the same place in dyadic algebra and tensor calculus as unity in ordinary scalar calculations.

It is outside the scope of this book to enter into further details of the dyadic algebra developed by Gibbs or that of matrices used in dealing with tensors. It will be sufficient to say that both form complete, parallel and equivalent systems, and have been applied to a wide variety of problems in pure and applied mathematics. They are of the non-commutative class in regard to certain products which, as with vectors, are matters of definition.

7. Application of Linear Vector Functions. It is a matter for personal preference whether one uses the tensor or dyadic notation in practice, though present tendencies would appear to show that tensors and matrices are nowadays the more usual. The simple cartesian tensors we have considered are fundamental in all problems of anisotropy, whether magnetic, electric, thermal or elastic, and also in certain purely geometric applications to kinematics and kinetics where transformations of rectangular axes are involved, as for example in the treatment of finite rotations.

More complex tensors, which we have no space to discuss, occur in the transformation of space-time coordinates from one observer to another in the theory of relativity. Also, in the general theory of electromagnetic machinery the properties of a number of electric circuits in relative motion are connected by sets of linear equations. The use of tensor and matrix notation is a natural means for investigating such problems as Gabriel Kron has recently shown with great success, but this very special application is entirely outside the purely vectorial scope of this volume and belongs properly to the general algebra of tensors.

POLAR CO-ORDINATES

THE most important operations of vector analysis have been expressed in the text in terms of components referred to rectangular or cartesian co-ordinates. Summarizing, they are

$$\text{grad } S = \nabla S = \frac{\partial S}{\partial x}\mathbf{i} + \frac{\partial S}{\partial y}\mathbf{j} + \frac{\partial S}{\partial z}\mathbf{k}\,; \qquad (4.3)$$

$$\text{div } \mathbf{V} = \nabla\cdot\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\,; \qquad (4.6,\ 4.8)$$

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k}\,; \qquad (4.9,\ 4.10)$$

$$\text{div grad } S = \nabla^2 S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} \qquad (5.1)$$

S is a scalar and $\mathbf{V}$ a vector point function. These usually suffice, but in some problems it is often more convenient to use some other system of co-ordinates specially adapted to the matter in hand; two of these will now be examined.

When there is symmetry about an axis, such as OZ, the cylindrical polar or columnar co-ordinates shown in Fig. 38 are appropriate. The position of a point P is specified by the polar co-ordinates r, θ and the axial distance z, these being related to the cartesian system by

$$x = r\cos\theta,\ y = r\sin\theta \text{ and } z \equiv z.$$

It is possible to transform the above operations from one system to the other by purely mathematical means, but it is more convincing to work out the operations afresh for the element of volume dr, $rd\theta$, dz shown to a larger scale in the lower part of the diagram, (r, θ, z) being the co-ordinates of its mid-point. As an example, if V_r be the radial component of a vector $\mathbf{V}$ at P the flux through the element in the r direction is

$$\left(V_r + \tfrac{1}{2}\frac{\partial V_r}{\partial r}dr\right)(r + \tfrac{1}{2}dr)d\theta\, dz - \left(V_r - \tfrac{1}{2}\frac{\partial V_r}{\partial r}dr\right)(r - \tfrac{1}{2}dr)d\theta\, dz,$$

which to the third order of small quantities is

$$\frac{\partial V_r}{\partial r}r\, d\theta\, dr\, dz + V_r d\theta\, dr\, dz.$$

Since the volume of the element is $r\,d\theta\,dr\,dz$, the radial component of div **V** is

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r).$$

The other components are found by a similar process. Likewise, by the method of pp. 43–5 the components of curl **V** can

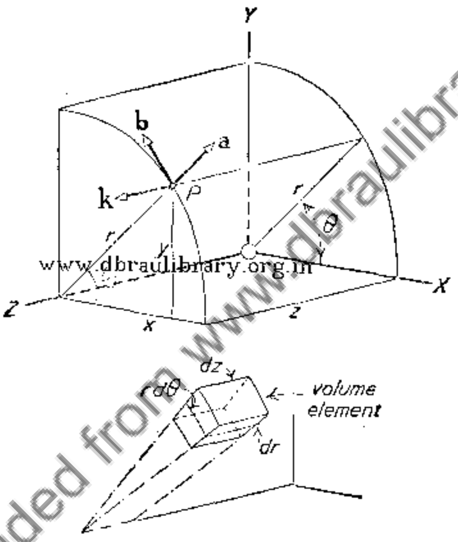


FIG. 38.—Cylindrical Polar Co-ordinates

be calculated; this is left to the reader as an exercise. Let **a**, **b**, **k** be unit vectors at P, **a** being in the positive direction of r, **b** in the sense of the positive tangent to the circle of radius r at P (i.e. along the arc $rd\theta$) and **k** along OZ. Note that **a**, **b**, **k** form a mutually perpendicular, right-handed system; **a** and **b**, though of unit magnitude, are functions of the position of P. Then, if V_r, V_θ, V_z are components of the vector point function **V** in the **a**, **b** and **k** directions respectively, it is easy to show that

$$\text{grad } S = \nabla S = \frac{\partial S}{\partial r}\mathbf{a} + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{b} + \frac{\partial S}{\partial z}\mathbf{k};$$

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z};$$

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right)\mathbf{a} + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\mathbf{b} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta}\right)\mathbf{k};$$

$$\text{div grad } S = \nabla^2 S = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \frac{\partial^2 S}{\partial z^2}.$$

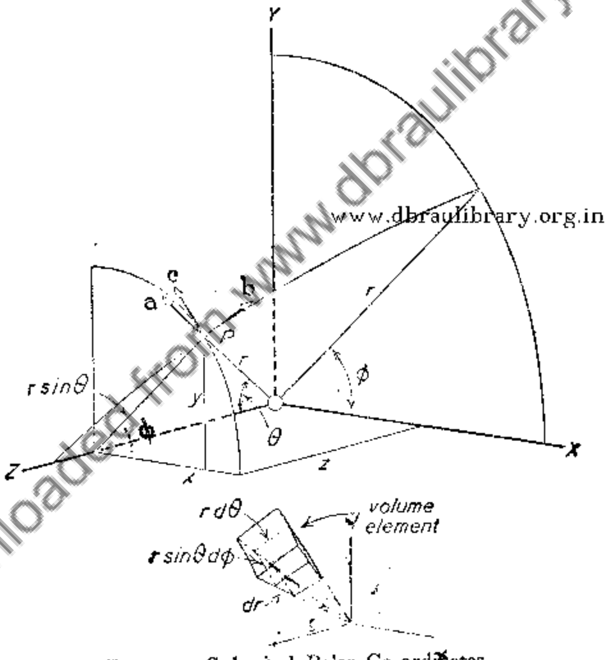


FIG. 39.—Spherical Polar Co-ordinates

When there is symmetry about a point, as in many geometrical and dynamical problems, the spherical polar co-ordinates shown in Fig. 39 are useful. The position of P is given by the radius

r, the angle of co-latitude between OZ and r, and the angle of longitude ϕ between the ZOX plane and that which contains OZ and r. The relation to the cartesian system is

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta.$$

Taking (r, θ, ϕ) as the co-ordinates of the middle of the volume element dr, $rd\theta$, $r\sin\theta d\phi$, let **a**, **b**, **c** be a right-handed orthogonal set of unit vectors at P in the positive directions of these increments; note that **a**, **b**, **c** are functions of the position of P. If V_r, V_θ, V_ϕ are components of a vector point function **V** in these directions, it is easy to prove for the volume element considered that

$$\text{grad } S = \nabla S = \frac{\partial S}{\partial r}\mathbf{a} + \frac{1}{r}\frac{\partial S}{\partial\theta}\mathbf{b} + \frac{1}{r\sin\theta}\frac{\partial S}{\partial\phi}\mathbf{c};$$

$$\text{div } \mathbf{V} = \nabla\cdot\mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi};$$

$$\text{curl } \mathbf{V} = \nabla\times\mathbf{V} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(V_\phi\sin\theta) - \frac{\partial V_\theta}{\partial\phi}\right]\mathbf{a}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial}{\partial r}(rV_\phi)\right]\mathbf{b} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right]\mathbf{c};$$

$$\text{div grad } S = \nabla^2 S = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2}.$$

PROPERTIES OF ∇ AS A FORMAL VECTOR

On p. 33 a definition of ∇ in cartesian vector form has been given, but it must be remembered that ∇ cannot be regarded as a true vector since $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ are differential operators and not true scalars. The symbol ∇ actually denotes an invariant vector differentiator and, as such, requires an operand, whether scalar or vector, upon which it can appropriately act. Its expression in vectorial form, therefore, can only be taken as *symbolic* or *formal*, inasmuch as ∇ behaves like a vector in formal associative and distributive differentiations such as

$$\nabla(U + W) = \nabla U + \nabla W$$

or

$$\nabla(UW) = (\nabla U)W + U(\nabla W)$$

where U and W are scalar point functions; these and many other simple operations the reader will find in the text-books and may easily verify.

It must not be assumed, however, that ∇ in more complex operations *necessarily* obeys all the rules of vector algebra, since ∇ is only formally and not actually a vector. Care must be taken to see that rigid application of those rules does not destroy the differentiative effect of the complete operation. Particular care is needed in using the commutative law; examples are found in the operations $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$ on a vector point function $\mathbf{V}$, which become meaningless when commutated (see pp. 42 and 45). We conclude, therefore, that ∇ is essentially and primarily an operator; secondarily, and then only as an analytical convenience, is it expressed in cartesian vector form. In arranging more complex operations the object should always be to choose such a use of the vector laws as will leave differentiative operations available to act on the operand.

An important practical case is that of the operator defined by $\mathbf{C} \cdot \nabla$ where

$$\mathbf{C} = C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k}$$

is a constant vector, i.e. one independent of x, y, z. Taking the scalar product gives

$$\mathbf{C} \cdot \nabla = C_x\frac{\partial}{\partial x} + C_y\frac{\partial}{\partial y} + C_z\frac{\partial}{\partial z},$$

which is a scalar differentiator applicable either to a scalar or to a vector point function. Note that $\nabla\cdot\mathbf{C}$ would vanish and thus have no meaning. In particular let

$$\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$$

be a vector of unit length in a given fixed direction defined by c_x, c_y, c_z, its components along the axes, i.e. by the direction-cosines of $\mathbf{c}$. Then the operation

$$\mathbf{c}\cdot\nabla = c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}.$$

Apply this operator to a scalar point function S, then

$$(\mathbf{c}\cdot\nabla)S = c_x\frac{\partial S}{\partial x} + c_y\frac{\partial S}{\partial y} + c_z\frac{\partial S}{\partial z}.$$

Now from Equation 4.3

$$\text{grad } S = \nabla S = \frac{\partial S}{\partial x}\mathbf{i} + \frac{\partial S}{\partial y}\mathbf{j} + \frac{\partial S}{\partial z}\mathbf{k},$$

is the greatest rate of increase of S at a point in magnitude and direction; and $\mathbf{c}\cdot(\text{grad } S)$ is the resolved or component magnitude of the rate of increase in the direction of $\mathbf{c}$, a scalar point function. Calculating this scalar product and comparing results shows that

$$(\mathbf{c}\cdot\nabla)S = \mathbf{c}\cdot(\text{grad } S) = \mathbf{c}\cdot(\nabla S).$$

This is known as the *directional derivative* of S in the direction of $\mathbf{c}$, expressing the magnitude of the rate of increase of S in this specified direction at any point. Similarly, for a vector point function

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k},$$

$$(\mathbf{c}\cdot\nabla)\mathbf{V} = (\mathbf{c}\cdot\nabla)V_x\mathbf{i} + (\mathbf{c}\cdot\nabla)V_y\mathbf{j} + (\mathbf{c}\cdot\nabla)V_z\mathbf{k},$$

where $$(\mathbf{c}\cdot\nabla)V_x = c_x\frac{\partial V_x}{\partial x} + c_y\frac{\partial V_x}{\partial y} + c_z\frac{\partial V_x}{\partial z},$$

is the directional derivative of $\mathbf{V}$ in the direction of $\mathbf{c}$; it is a vector point function.

This theorem provides an alternative proof of a result given on p. 47, namely, to calculate $\nabla \times \boldsymbol{\omega} \times \mathbf{r}$, where

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

is a constant vector and

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

is a variable one. Expanding as a vector triple product by Equation 2.22,

$$\nabla \times \boldsymbol{\omega} \times \mathbf{r} = (\nabla \cdot \mathbf{r})\boldsymbol{\omega} - (\nabla \cdot \boldsymbol{\omega})\mathbf{r}$$

But

$$\nabla \cdot \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

Now from above $\nabla \cdot \boldsymbol{\omega}$ has no operational meaning; we interpret it, therefore, as $\boldsymbol{\omega} \cdot \nabla$. Substituting $\boldsymbol{\omega}$ for $\mathbf{C}$,

$$\boldsymbol{\omega} \cdot \nabla = \omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z},$$

and

$$(\boldsymbol{\omega} \cdot \nabla)\mathbf{r} = \left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k} = \boldsymbol{\omega}.$$

Finally,

$$\nabla \times \boldsymbol{\omega} \times \mathbf{r} = 3\boldsymbol{\omega} - \boldsymbol{\omega} = 2\boldsymbol{\omega},$$

as found by direct expansion on p. 47.

BIBLIOGRAPHY

The Bibliography of Vector Analysis is very extensive and it is not possible to do more than refer to a few of the books likely to be useful to the reader of this monograph. J. G. Coffin, *Vector Analysis. An introduction to vector methods and their various applications to physics and mathematics* (Wiley, 1911); C. E. Weatherburn, *Elementary Vector Analysis* (Bell, 1926) and *Advanced Vector Analysis* (Bell, 1928); and D. E. Rutherford, *Vector Methods applied to differential geometry, mechanics and potential theory* (Oliver and Boyd, 1939), may be specially recommended. All these volumes use the same notation as this monograph and are liberally provided with exercises to enable the student to test his progress. The standard classic on the subject is E. B. Wilson, *Vector Analysis* (Scribner, 1902), based directly on the lectures of J. W. Gibbs, but its encyclopaedic scope makes it more valuable for reference than for continued study. In German there are many treatises, one of the most complete being J. Spielrein, *Vektorrechnung* (Wittwer, 1926). Excellent smaller works are R. Gans, *Vektoranalysis mit Anwendungen auf Physik und Technik* (Teubner, 1929), published also in English by Messrs. Blackie; S. Valentiner, *Vektoranalysis* (de Gruyter, 1929); and H. Schmidt, *Einführung in die Vektor- und Tensor-rechnung* (Jänecke, 1935). The properties of Cartesian tensors are fully developed by H. Jeffreys, *Cartesian Tensors* (Cambridge University Press, 1931). An excellent elementary account of matrix algebra is given by C. V. Durell and A. Robson, *Advanced Algebra*, Vol. 3 (Bell, 1937).

Among books using vector methods the writings of O. Heaviside are classic, but of considerable difficulty. The reader may obtain many useful ideas from Vol. 1 of *Electromagnetic Theory*. M. Abraham and A. Föppl, *Einführung in die Maxwellsche Theorie der Elektrizität* (Teubner, 1907) is specially to be recommended. G. Joos, *Theoretical Physics* (Blackie, 1934) covers the whole range of mathematical physics by exclusively vectorial methods and is an excellent book for the more advanced reader.

NOTATION

The notation used in this book is that of Gibbs, with the additional adoption from Lorentz of the term 'grad' to represent the differentiation of a scalar by the operator ∇. On the Continent certain differences of notation occur and these also appear in some English books (often translations). These are (i) the use of **AB** instead of **A·B**, or the employment of round brackets (**AB**) to denote the scalar product. (ii) The use of [**AB**] for the vector product **A**×**B**. (iii) The substitution of 'rot' for 'curl'. (iv) The writing of **A.B** or of **A**; **B** for the dyad **AB**. These differences are easily remembered and vector equations can be read with equal facility either in the Gibbs or the Continental system after a little practice.

Heaviside uses an unsymmetrical notation that makes his writings unnecessarily hard to follow; he uses **AB** and *V***AB** for the scalar and vector products respectively.

In reading vector equations, expressions such as **A·B** and **A** × **B** are read 'A dot B' and 'A cross B' respectively. Similarly, ∇S is read 'del S', ∇·**V** is 'del dot **V**' and ∇ × **V** is 'del cross **V**'.

The notation for line, surface and volume integrals follows Gibbs, i.e.

$$\int(\quad)dl,\ \iint(\quad)ds \text{ and } \iiint(\quad)dv$$

respectively, the number of integral signs being a useful reminder that dl has one, ds two and dv three dimensions of length irrespective of any particular co-ordinate system.

Historically the terms used in vector analysis are of great interest. Although the operator ∇ was introduced by Hamilton he left its development to his disciple P. G. Tait, who adopted the name 'nabla' for this symbol, following a suggestion of W. Robertson Smith. Maxwell suggested 'atled', but the name 'del' due to Gibbs is preferred for shortness and euphony. To Maxwell are due the terms 'slope' (for 'grad'), 'convergence' (for − 'div'), 'curl' and 'concentration' (for ∇^2).

NOTE ON MAXWELL'S EQUATIONS

THE deduction of Maxwell's equations for the electromagnetic field given on p. 92 *et seq.* follows that given in Chap. IX, Vol. II of his *Treatise on Electricity and Magnetism*, all quantities being expressed in electromagnetic units. It is very common practice, however, to state **c**, **H**, μ and **B** in the electromagnetic system, but $\mathfrak{E}$, κ and **D** in the electrostatic system. If v is the velocity of light in free space the last three quantities are expressed in electromagnetic units by writing $v\ \mathfrak{E}$, κ/v^2 and $\mathbf{D}/v$. Equation 9.2 then becomes

$$\mathbf{C} = \mathbf{c} + \frac{1}{v}\dot{\mathbf{D}} = \mathbf{c} + \frac{\kappa}{4\pi v}\dot{\mathfrak{E}} ;$$

and the field Equations 9.4 and 9.5 are

$$\text{curl } \mathbf{H} = 4\pi\mathbf{C} = 4\pi\mathbf{c} + \frac{\kappa}{v}\dot{\mathfrak{E}}$$

$$\text{curl } \mathfrak{E} = -\frac{1}{v}\dot{\mathbf{B}} = -\frac{\mu}{v}\dot{\mathbf{H}},$$

together with

$$\text{div } \mathbf{B} = 0$$

and

$$\text{div } \mathbf{D} = \rho$$

The reader will find it a useful exercise to work out any modifications introduced into Equations 9.6 to 9.13 by the use of this mixed system of units.

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